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SELECTED THEORETICAL AND PRACTICAL ASPECTS OF STUDIES MADE IN CONJUNCTION WITH THE JOINT CANADA/FRG RESEARCH PROJECT ON COARSE-SLURRY, SHORT-DISTANCE PIPELINING

L.B. Geller and W.M. Gray
Canadian Mine Technology Laboratory

MINERAL & ENERGY TECHNOLOGY
MINING RESEARCH LABORATORIES

SP 86-16E

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DECEMBER 1986

MINERAL & ENERGY TECHNOLOGY
MINING RESEARCH LABORATORIES

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SP86-16E

CANADA AWARDS FOR EXCELLENCE 1986

SILVER AWARD IN TECHNOLOGY TRANSFER

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Available in Canada through

Associated Bookstores
and other booksellers

or by mail from

Canadian Government Publishing Centre
Supply and Services Canada
Ottawa, Canada K1A 0S9

Catalogue No. M38-15/86-16E Canada: $31.50
ISBN 0-660-12631-1 Other Countries: $37.80

Price subject to change without notice

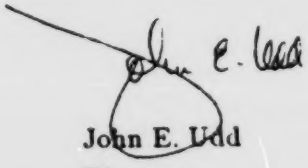
FOREWORD

Within the general scope of a 1971 agreement between Canada and the Federal Republic of Germany (FRG) on scientific and technical cooperation, a Memorandum of Understanding was signed in 1984 by CANMET and the Ministry of Research and Technology (BMFT) of the FRG concerning the study of coal-slurry systems.

This report represents a significant output of that study on the Canadian side. In it, the authors study some aspects of the Wilson model for slurry flow, including its connection with the formulae of Durand and of Newitt et al. Some comparative calculations are given to illustrate the equivalence of Wilson's and Shook's use of the force balance equations, as well as short studies of some theoretical and practical topics relating to the Wilson model. Moreover, a plan of data accumulation, based on dimensional analysis, is also proposed.

Because it may be of interest to many professionals, it has been decided to issue the report in a form which is suited to a broad distribution.

The authors are to be commended for their efforts in writing the document. In the case of Dr. Gray, this has represented a post retirement labour of love. We, at MRL, are grateful for this commitment.



John E. Udd
Director
Mining Research Laboratories

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AVANT-PROPOS

Dans le cadre d'un accord de coopération scientifique et technique conclu en 1971 entre le Canada et la République fédérale d'Allemagne (RFA), un mémoire d'entente a été signé en 1984 entre le CANMET et le ministère de la recherche et de la technologie (BMFT) de la RFA concernant l'étude de systèmes de transport des bouillies de charbon.

Le présent rapport constitue une part importante de la contribution canadienne à cette étude. Il comprend une étude de certains aspects du modèle de Wilson pour l'écoulement des bouillies, y compris sa relation avec la formule de Durand et celle de Newitt et al. Les auteurs donnent certains calculs comparatifs afin de démontrer que les équations de l'équilibre des forces utilisées par Wilson et par Shook sont équivalentes et exposent des études sommaires sur des sujets théoriques et pratiques relativement au modèle de Wilson. De plus, ils proposent un plan d'accumulation des données fondé sur l'analyse dimensionnelle.

Étant donné que le rapport pourra intéresser de nombreux professionnels, il sera publié sous une forme qui permettra une grande diffusion.

Les auteurs du rapport méritent des félicitations pour leur travail de rédaction. En ce qui concerne M. Gray, ce document représente une oeuvre chère à laquelle il s'est dévoué après sa retraite. L'équipe des Laboratoires de recherche minière lui est très reconnaissante de son engagement.

John E. Udd
Directeur
Laboratoires de recherche minière

ABSTRACT

This report is part of the contribution by the Canada Centre for Mineral and Energy Technology (CANMET) under a Memorandum of Understanding on Scientific and Technical Cooperation signed in 1984 by Canada and the Federal Republic of Germany.

The model proposed by K.C. Wilson for coarse-slurry pipelining is central to the work reported on. A study of some aspects of the model, including its connection with the formulae of Durand and of Newitt et al., concludes that the model should be treated as a new formulation for the empirical correlation of data. Difficulties involved in considering the parameters of the model as physical properties are discussed.

Comparative calculations relating to the Wilson model illustrate the equivalence of two different systems of equations for the model, presented by K.C. Wilson and by C.A. Shook, under the same basic assumptions. A connected study elucidates a difference between the treatments by K.C. Wilson and by L.L. Eyler et al. of the interfacial friction factor in the model. It is also shown that a correct solution of the equations of the model is highly sensitive to the value of the interfacial friction factor, although the value of the hydraulic gradient is not. A simple means is given for fitting Wilson's parameter ζ to a set of data, if all other parameters are known.

A plan for data accumulation, based on dimensional analysis, is proposed. An introduction to dimensional analysis for slurry flow in pipes is appended.

The results of short studies of some theoretical and practical topics encountered in the literature relating to the Wilson model are presented.

ACKNOWLEDGEMENT

The authors gratefully acknowledge the kind cooperation received during the course of a number of personal meetings and discussions with Professor K.C. Wilson of Queen's University, Kingston, Ontario, and with Professor C.A. Shook, of the University of Saskatchewan, in Saskatoon.

RÉSUMÉ

Le présent rapport constitue un des éléments de la contribution du Centre canadien de la technologie des minéraux et de l'énergie (CANMET) dans le cadre d'un mémoire d'entente de coopération scientifique et technique, signé en 1984, entre le Canada et la République fédérale d'Allemagne.

Le modèle, proposé par K.C. Wilson concernant le transport par pipeline des bouillies de charbon, forme la partie centrale des travaux décrits dans ce rapport. Une étude de certains aspects du modèle, y compris sa relation avec la formule de Durand et celle de Newitt et al, permet de conclure que le modèle devait être traité comme une nouvelle formulation pour faire la corrélation empirique des données. Les auteurs discutent des difficultés soulevées lorsque les paramètres du modèle sont considérés comme des propriétés physiques.

Des calculs comparatifs se rapportant au modèle de Wilson démontrent que les deux différents systèmes d'équations sont équivalents pour le modèle présenté par K.C. Wilson et par C.A. Shook, à partir des mêmes hypothèses fondamentales. Une étude connexe fait ressortir une différence entre la façon de K.C. Wilson et celle de L.L. Eyler et al de traiter le coefficient de frottement interfacial dans le modèle. Il est également établi qu'une solution correcte des équations du modèle est fortement influencée par la valeur du coefficient de frottement interfacial, bien que la valeur du gradient hydraulique ne le soit pas. Le rapport offre un moyen simple de faire correspondre le paramètre de Wilson ζ à une série de données, si tous les autres paramètres sont connus.

Les auteurs proposent un plan d'accumulation des données fondé sur l'analyse dimensionnelle. Une introduction à l'analyse dimensionnelle de l'écoulement des bouillies dans les canalisations figure en annexe.

Le rapport présente les résultats d'études sommaires sur certains sujets théoriques et pratiques qui ont été traités dans les ouvrages relativement au modèle de Wilson.

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LIST OF SYMBOLS

A	area of cross section of a pipe, or symbol for a general property of a flow
A'	*prototype* value of some property A
A''	*model* value of property A
A_1	area of sector 1 (upper sector of pipe cross section)
A_2	area of sector 2 (lower sector of pipe cross section)
A_{12}	part of A_1 that is influenced, hypothetically, by the bed surface bordering it
A_w	part of A_1 that is influenced, hypothetically, by the pipe wall bordering it
a	$= A_1/A$ dimensionless area of sector 1
C, C_t	average volumetric concentration of particles in a mixture flowing in a pipe in situ (Wilson and Brebner write C_v)
C_1	average volumetric concentration of particles in sector 1
C_2	average volumetric concentration of particles in sector 2
C_1'	$= C_{f12}/C_{fi}$ a factor due to the effect of particles saltating on the surface of the bed (Eyler writes C_1)
C_b	the average in situ volumetric concentration of particles in a loose-packed bed
C_c	contact-load in situ, the part of C that is supported by the pipe invert
C_D	drag coefficient of a particle in fluid, far from any boundary or other particles
C_d	average volumetric concentration of particles in a delivered mixture (Condolios and Chapus write C_t; Wilson and Brebner write C; Babcock; Zandi; Hayden and Stelson write C_v)
C_{dc}	delivered contact-load, the part of C_d that is supported by the pipe invert in transit
C_{dh}	delivered suspended load, the part of C_d that is supported hydraulically in transit
C_f	total hydraulic friction factor (Fanning) of flow in a pipe

C_{f0}	hydraulic friction factor for a flow of clear water
C_{f1}	hydraulic friction factor characteristic of pipe wall roughness for flow in sector 1
C_{f2}	hydraulic friction factor characteristic of pipe wall roughness for flow in sector 2
C_{f12}	effective fricion factor of the bed surface
C_{fi}	friction factor of the bed surface, with all particles fixed to the bed (Eyler)
C_r	$= C_d/C_b$
C_w	average concentration by weight, corresponding to volumetric concentration C
c	local volumetric concentration of particles in a mixture flowing in a pipe (in situ), or
c	$= V/v_* = \sqrt{8/f}$, a dimensionless variable representing V
D	inside diameter of pipe
d	characteristic particle diameter of a class of particles
d_i	characteristic d of particle fraction i
f	$= 4\,C_f$ hydraulic friction factor (Darcy-Weisbach), or
	friction factor given by the Nikuradse formula
f'	friction factor for a flow of water in a clear pipe at the same mean velocity
	as a slurry flow in the same pipe (Wilson model)
$f_A(\ldots)$	a function that determines the property A
g	acceleration due to gravity
i	hydraulic head gradient of a fluid, or slury, in metres of water per metre of pipelength
i_1, i_2	hydraulic gradients applying to sectors 1 and 2, respectively (Wilson model)
i_w	hydraulic gradient of a flow of clear water
j_p	$= 2\,\eta\,(s-1)\,C_b$ minimum hydraulic gradient required to maintain plug flow
	of a loose-packed bed of particles that fills the pipe (Wilson model)
k	roughness height characteristic of a pipe wall inner surface
m	moisture concentration (by weight) in a water-saturated solid, or a positive constant
N	number of size fractions of a particle-size distribution
P	pressure, in fundamental units
R_{ep}	$= w\,d/\nu$ Reynolds number for terminal fall velocity

S area of pipe wall, per unit length of pipe

S_1 area of pipe wall, per unit length of pipe, bordering sector 1
(also used to designate the surface)

S_2 area of pipe wall, per unit length of pipe, bordering sector 2
(also used to designate the surface)

S_{12} area of interface between sectors 1 and 2, per unit length of pipe
(also used to designate the surface)

s $= \rho_s/\rho$ ratio of solid-to-carrier fluid densities
(fluid is usually water)

s_s $= s$

t time

u_* $= \sqrt{i\,g\,D/4}$ shear, or friction, velocity of a flow in a pipe

V average velocity of carrier fluid and of particles (in the whole pipe)

V_1 average velocity in sector 1

V_2 average velocity in sector 2

V_C velocity at the minimum i of an $i - V$ curve

V_R $= V_2/V_1$

V_t the velocity at which partial suspension of particles commences as V is increased

v_* $= u_*$

W $= X_4$

W_f weight of free water in a sample of a slurry

W_s weight of saturated particles in same sample as for W_f

w terminal fall velocity of a particle in water

w_i fall velocity characteristic of a particle of size-fraction i

X $= \frac{i - i_w}{(s-1)\,C_d}$ (see section 'Contact-load to Total-load Relationships'), or

X $= X_1$

X_1 $= X = v_*\,d/\nu$ dimensionless variable (see Appendix B Table (B-2))

X_2 $= Y = \frac{\rho v_*^2}{\gamma_* d}$ dimensionless variable (see Appendix B Tables (B-2) and (B-3))

X_3 $= Z = D/d$ dimensionless variable (see Appendix B Tables (B-2) and (B-3))

X_4 $= W = \rho_*/\rho$ dimensionless variable (see Appendix B Tables (B-2) and (B-3))

X_5 $= C$ dimensionless variable (see Appendix B Tables (B-2) and (B-3))

X_6 $= k/d$ dimensionless variable (see Appendix B Tables (B-2) and (B-3))

X_7 $= \eta$ dimensionless variable (see Appendix B Tables (B-2) and (B-3))

$\overline{\overline{X}}$ $= \Xi = \frac{\gamma_* D^3}{\rho \nu^2}$ dimensionless variable (see Appendix B Table (B-3))

Y $= X_2$

Z $= X_3$

(x, y, z) Cartesian co-ordinates, z in direction of flow, y directed vertically up

β value of θ for the intersection of the pipe wall with the surface of the bed

γ $= g \rho$ specific weight of fluid (usually water)

γ_* $= g (\rho_* - \rho)$ specific net weight of a particle submerged in fluid

ϵ roughness height of a bed surface (Eyler)

ζ $= C_{f12}/C_{f0}$ (Wilson)

η coefficient of mechanical sliding friction between particles and pipe

θ angle in cylindrical co-ordinate system centred on a pipe axis, measured from the negative y-axis

λ_A $= A''/A'$ scale factor for property A

μ viscosity of a fluid (usually water)

μ_1 viscosity of a slurry consisting of a fluid and suspended particles

ν $= \mu/\rho$ kinematic viscosity of a fluid

Ξ $= \overline{\overline{X}}$

ζ $= C_{fi}/C_{f0}$ (Eyler) $= \zeta/C_1'$

Π_A dimensionless version of property A, with basic parameters d, ρ, v_*

$\overline{\overline{\Pi}}_A$ dimensionless version of property A, with basic parameters d, ρ, γ_*

ρ, ρ_f density of a fluid (usually water)

ρ_1, ρ_2 average densities of mixtures in sectors 1 and 2, respectively

ρ_d average density of a total delivered mixture of water and particles

ρ_m, ρ_{sl} average density of a slurry or mixture

ρ_s density of non-porous solid particles, or average density of
of porous solid particles when saturated with water

ρ_{ss} true density of the non-porous solid material forming part of a porous material

ρ_w density of water

σ normal stress, or characteristic width of a particle-size distribution

σ_{rs} pressure normal to surface S_2 at the level specified by θ,
due to the submerged weight of contact particles

τ shear stress

τ_0 shear stress at the pipe wall, due to a flow of clear water in the clear pipe

τ_1 shear stress on surface S_1

τ_{12} shear stress on surface S_{12}

$\overline{\tau}_2$ shear stress on surface S_2

τ_{2f} component of $\overline{\tau}_2$, due to fluid flow

$\tau_{2\eta}$ component of $\overline{\tau}_2$, due to mechanical sliding friction

$\Phi_A(\ldots)$,

$\overline{\overline{\Phi}}_A(\ldots)$ function of the characteristic dimensionless variables of a flow
that determine Π_A and $\overline{\overline{\Pi}}_A$, respectively

ϕ $= \dfrac{i - i_w}{C_d\, i_w}$

ϕ_t $= \dfrac{i - i_w}{C_t\, i_w}$

$\phi_1(\beta)$ $= \dfrac{\sin\beta - \beta\cos\beta}{\pi}$ (Wilson model)

ψ $= \dfrac{V^2\sqrt{C_D}}{g\,(s-1)}$

Selected Theoretical and Practical Aspects of Studies Made in Conjunction with the Joint Canada/FRG Research Project On Coarse-Slurry, Short-Distance Pipelining

by

L.B. Geller* and W.M. Gray**

BACKGROUND

An agreement was signed in 1971 by Ministers Jean-Luc Pepin and Walter Scheel, for the governments of Canada and the Federal Republic of Germany (FRG), respectively, on scientific and technical cooperation (see Appendix A of Geller and Prof. Geller, 1984). Its purpose was:

- to broaden the scope of all aspects of scientific and technological cooperation between the two countries, for peaceful purposes and for their mutual benefit; and
- to facilitate and to encourage scientific and technological cooperation and exchanges of information.

One special field of cooperation agreed to under the terms of this document was that of coal-slurry systems. Details of the proposed cooperation are covered by a Memorandum of Understanding (MOU), signed in 1984 by Dr. W.G. Jeffery for CANMET in Ottawa, and by Dr. Finke for the Ministry of Research and Technology (BMFT) in Bonn (see Appendix A of Geller, 1985). The four specific areas referred to in this MOU are:

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— dense coal-slurry systems, including their preparation, transportation, combustion, and marketing and commercial aspects;

— hydraulic transportation of coking coals;

— Coarse-slurry, short-distance pipelining; and

— preparation of an engineering manual on the design of slurry pipelines.

The project leaders, identified by the MOU for the slurry pipelining research area, are L.B. Geller of CANMET's Mining Research Laboratories in Ottawa, and Professor F.J. Geller of the Westfälische Berggewerkschaftskasse's (WBK's) Mining Engineering Department in Bochum. They were charged with preparing a report detailing the objective, methodology, costs, and timetable of a mutually acceptable, joint R&D project on coarse-slurry, short-distance pipelining. Such a program was duly developed and reported on (Geller and Prof. Geller, 1984). It was submitted to the Canadian and German coordinators designated by the MOU (Dr. T.D. Brown and F. Fiseni, respectively), and accepted by them. To paraphrase Professor Shook, the goals, rationale, and approach adopted for this program were, essentially, as follows:

Goals :

To provide a method for determining the hydraulic performance of

— run-of mine coals
— cleaned coals
— washery wastes

when transported by pipeline for comparatively short distances.

Rationale :

In contrast to long-distance transport of fine-particle slurries, pipelines for short-distance movement of coarse solids cannot justify the heavy expenses of thorough testing programs, before they are designed. Instead, designers must rely on a model, and on a minimal amount of data.

Approach :

In an independent US Department of Environment study (Eyler et al., 1982), empirical models were shown to be less satisfactory than mechanistic ones. In particular, the force/mass-balance model of Wilson (itself an elaboration of that of Newitt et al., 1955) was identified as the most suitable one for use. Wilson's model was derived for use with coarse-slurry flow, but the data upon which it was based involved only coarse sand in small pipes, at low concentrations. Also, the values of d/D were no greater than 0.03. The range of operating conditions must, therefore, be extended, before it can be adopted for a full range of design purposes.

Because western Canadian coal is very friable, the effect of particle diameter in the model is of obvious interest in Canada. Therefore, in its first phase, the Canadian study needs to address this problem, with particular attention to such parameters as pipe diameters and slurry viscosities.

As for the situation in West Germany, it is known that a number of studies have already been conducted there with very coarse coal and waste materials. However, it is impossible to incorporate these data in the model, unless the conditions of the relevant experiments can be fully established. In particular, the viscosity of the fines-containing fraction, as well as the coefficient of particle-to-pipe wall friction must be established. It is also desirable that additional tests be conducted in Germany, to complement the ones scheduled for Canada. This is of particular interest in areas for which the test equipment at the StBV laboratory in Essen is better suited for certain of the experiments, than is SRC's in Saskatoon, e.g., in the particle range above about 6 mm. In summary, it is hoped to set up a combined Canada/FRG research effort that will yield a model, which can then be confidently used for a full range of design purposes.

The opportunity provided by the Canada/FRG joint research agreement is an unusually auspicious one for setting up an in-depth test series with coarse solids, on a carefully controlled scientific basis, in a range of pipe sizes, with various types of materials. Such

conditions usually are not possible in testing, when only practical, short-term results are sought. The resulting data, then, cannot be included, with any degree of confidence, in the data bank of others. Moreover, the test results may be of a proprietary nature, and cannot, therefore, augment general knowledge. Under the Canada/FRG agreement it is possible to ensure that all relevant results are widely disseminated, for the benefit of both research and industrial interests in general.

As a consequence of this joint research agreement, an appropriate test contract was devised and initiated by CANMET, as Canada's initial project contribution. Preliminary results achieved from this contractual effort, and their evaluation with a mechanistic model, have been given elsewhere (Gillies et al., 1985, 1986; Shook et al., 1986). Moreover, the relevance of these test results, within the overall framework of the cooperative project, as well as their immediate practical significance, was also documented elsewhere (Geller, 1985). This documentation includes both specific operational control problems, as well as general implications regarding the overall industrial relevance of slurry research.

Besides the previously mentioned laboratory test results, Canadian members of the joint Canada/FRG 'Coarse-Slurry Working Group' contributed a number of theoretical essays to the program. These reports deal, in particular, with the modelling aspects of interest, including several prepared by Professor C.A. Shook (Shook, 1980; 1981; and 1983). CANMET scientists also contributed reviews of several theoretical aspects, primarily by means of live discussions and through unpublished communications. This report is intended to serve as an appropriate documentation of the latter and as a contribution to a useful exchange of ideas.

Although German members of the cooperative project have, so far, been unable to proceed with any of their own laboratory work, because of funding problems, they too have tabled essays dealing with some of the modelling aspects in question (e.g., Hartbrich, 1984 and 1985; Nacke and Verholen, 1984).

INTRODUCTION

Hydraulic systems have been used for many years to transport solids, usually in the form of fine-particle slurries, but some installations have also been used to move rock in large-sized pieces. Bain and Bonnington (1970) mention examples in which sizes ranged up to 12 cm in diameter, as well as one case in which phosphate rock was moved in 20- to 25-cm pieces, for distances of several kilometres. However, experimentation relative to the improvement of efficiency is more difficult in case of coarse particles, than in case of fine slurries. Consequently, the knowledge of how to improve large-particle transportation has not developed quickly. Reviews of the general subject have been published by Zandi (1971) and by Shook (1976).

Apart from theoretical-cum-laboratory work with smooth-bore pipes, two rather special endeavours were also described in the literature, undertaken in an effort to come to grips, in pragmatic terms, with the transport of coarse-solids slurry. In one case, a helical rib is installed on the inner surface of the pipe, to give the slurry a rotary motion as it is pumped through the pipe. Experimental investigations at the Saskatchewan Research Council's Pipeline Development Centre have indicated that substantial savings in the transportation energy (of the order of 42%, according to one particular set of data) may be possible (Shriek et al., 1974; Shook, 1976).

In the case of the second special approach, the particulate material is encased in a cylindrical capsule, to be carried through the pipe as a unit. Experiments in the laboratories of the Alberta Research Council showed that, in this case also, savings in transportation energy might be realized. Numerous papers and reports have been issued; the results have been summarized by Shook (1976).

Other special solutions to slurry pipelining have also been studied, e.g., the method known as *Pulsing Flow* (Round, 1974, 1981, 1986). Despite a number of claims, to have

achieved savings in transportation energy by means of special methods, none have, to date, been accepted in general practice.

A new theoretical approach to modelling of hydraulic transport has been described recently by Roco and Shook (1981, 1982). The authors attempt to solve basic problems semi-empirically by extensive use of computer calculations. Results reported on initial applications of the method indicate that much improved accuracy and reliability may be achieved in the prediction of field performance from laboratory pipeline tests. However, by the very nature of hydraulic theory, various difficulties have not yet been solved in this approach. The ability to obtain results economically from a limited amount of data on a new particulate solid remains to be established.

Much of the work for the present report has been concentrated on developments, stemming principally from the introduction by K.C. Wilson (1976) of a, so-called, mechanistic approach to the mathematical modelling of slurry transportation. This method aroused considerable interest because of the conclusion that the Wilson model promised to be more satisfactory than previous empirical models (Eyler et al., 1982; Eyler and Lombardo, 1980; Lombardo and Eyler, 1980a, 1980b). To a certain extent Wilson and others (who will be mentioned later) achieved a better qualitative understanding of the mechanics of the transport process. However, progress in improvement of the model has proved to be difficult.

The first section of the report reviews some ideas, used for empirical correlations, that have influenced the development of the Wilson model. The second section discusses the general nature of the model. Close consideration of the divergence of the conditions of a real flow from the basic assumptions of the model leads to the conclusion that its parameters cannot be treated as physical properties that can be measured independently. The hope remains, however, that, as an improved formulation, it may perform more reliably on a minimum amount of data for a given area of interest, than earlier correlation systems.

The third section describes the basic assumptions of a simple version of the Wilson model. Equations are given in accordance with Wilson's approach (referred to as Model A)

and in accordance with the approach of Shook et al. (referred to as Model B). Calculations illustrate the equivalence of the two models. Some differences between these models and the approach of Eyler et al. are also discussed. A simple method of solution of the model is given for the case where the interfacial shear stress parameter, Wilson's ζ, is to be determined, all other parameters being fixed.

It is generally agreed that suitable data are sorely needed for further development of the scientific basis for coarse-particle slurry transport. In this context see Eyler et al. (1982), Shook (1976, p. 21), Shook et al. (1981, p. 91), and Shook (1983). It is also generally accepted that accumulation of such data is a difficult and expensive undertaking and that, even when support is available, it normally applies only to the investigation of a given material under specific conditions. Appropriate harmonization of basic research and of field operations (including a careful scrutiny of the overall economic picture) must, therefore, never be overlooked.

Because of this situation a discussion is included, in the fourth section of the report, of the value of dimensionless analysis in planning experiments required to establish, as efficiently as possible, the data required for the development and use of any model. An exploratory proposal is made for a plan for accumulating a bank of data that would be suitable for reliable communication between laboratories.

It is also shown that dimensionless analysis can assist in providing short-term results before a good model is available. Due to the extreme complexity of the physical processes involved in hydraulic flow, an approach may have to be adopted that circumvents presently unsolved problems of modelling, if practical goals are to be achieved.

The fifth section of the report treats a number of varied topics encountered in the literature relating to the Wilson model. It includes discussions of theoretical problems, including applications of dimensional analysis, problems of practical experimentation, and problems of definition of symbols.

Appendices A, B, and C, respectively, present a very brief sketch of some basic features of fluid flow, an introduction to dimensional analysis for flow in pipes, and an application of dimensional analysis.

DISCUSSION OF EMPIRICAL CORRELATIONS USED IN PIPELINE DESIGN PROCEDURES

The Wilson model for slurry flows was developed to overcome defects in empirical equations, found by data correlation, for use as bases for pipeline design. Because some concepts used for the development of empirical correlations were adapted for use in later models, these are reviewed briefly. Certain observations, made in connection with the experimental work reported, have interesting implications for modelling.

Bain and Bonnington (1970) give a concise account of the development of the Durand equation, probably the best-known relation used for the correlation of slurry flows. Zandi (1971) reviews empirical relationships.

The Durand equation can be written as

$$\phi = K\psi^{-m} \qquad Eq\ 1$$

with

$$\phi = \frac{i - i_w}{C_d i_w} \qquad Eq\ 2$$

and

$$\psi = \frac{V^2\sqrt{C_D}}{g(s-1)D} \qquad Eq\ 3$$

where:

- i is the hydraulic head gradient in terms of water density, for a slurry flow with mean velocity V,
- i_w is the hydraulic gradient in terms of water density for a flow of clear water with mean velocity V, and
- K, m are constants. For other symbols, please refer to the symbol list, and to the referenced articles.

This correlation was developed for slurry flows in which the particles are not wholly held in uniform, or homogeneous, suspension. Such flows are termed *heterogeneous*. For sands and gravels ($s = \rho_s/\rho = 2.65$) particles greater than about 0.15 mm in diameter are subject to heterogeneous flow conditions.

Sizes less than 0.15 mm in diameter are referred to as *fine* sizes; sizes greater than 1.5 mm, as *coarse* sizes; and sizes between 0.15 and 1.5 mm, as *intermediate* sizes. The Durand equation is an attempt to deal with slurries involving particle sizes greater than about 0.15 mm.

Durand and his associates determined $m = 3/2$. Various values of K have been attributed to Durand, but values of K and m have both been subjects of discussion and uncertainty in the literature. However, the importance of the equation lies in its general form. The Durand equation need not be considered as a model with fixed values of K and m. If the latter are determined to fit the data for a specific particulate material, a useful model can be found, within the ranges of parameters tested.

In Situ Concentration, C_t, as a Parameter

As a variant of the Durand equation, Shook et al. (1981)† have substituted the in situ concentration of particles in the pipe, C_t, for the delivered concentration, C_d. The modified equation may be written as

$$\phi_t = K\psi^{-m} \qquad Eq\ 4$$

with

$$\phi_t = \frac{i - i_w}{C_t i_w}.$$

Shook et al. (1981) found that this equation can express reasonable agreement among the results observed with one single type of coal, using different pipe sizes. They found different K values with two different types of coal, and found $m = 1$ to fit their results. They carried out runs by recirculating the slurry. As the mean velocity was varied, C_t remained constant, and C_d varied with V. As V decreases, $\frac{i - i_w}{i_w}$ increases, and C_d decreases. Thus ϕ increases at a greater rate than ϕ_t, which was offered as an explanation for the value of $m = 1$, rather than the Durand value of $m = 3/2$.

The use of C_d for correlation is widespread, and often is taken for granted, without an explicit definition of concentration. In an interesting paper, Babcock (1971) almost certainly uses C_d (represented by him as C_v), obtained by sampling of the slurry flow; recirculation is not mentioned. Babcock found that the original Durand equation, with $m = 1$, gave excellent fits to his experimental observations with *intermediate* and *coarse* sands. This result seems to imply a discrepancy between Babcock (1971) and Shook et al. (1981), because both found $m = 1$, although one used C_d, and the other C_t. Differences between the parameters of the flows in the two cases, not taken into account by the Durand equation, may be the cause of the discrepancy.

† Note that in Figures 6, 7, 8, and 9 of the referenced paper, the axis of abscissae should be labelled ψ, and not ψ^{-1}, according to the common definition of ψ.

Coefficient of Mechanical Friction, η, as a Parameter

Mechanical friction is used to indicate friction resulting from sliding displacement between solids in contact, as distinct from viscous friction.

Investigators have, for a long time, realized that, when particles were not carried in homogeneous suspension, mechanical friction between the particles and the wall of the pipeline could contribute to the head losses of the flow, in addition to viscous friction. $(i-i_w)$ was seen as the increment of the gradient due to solids in the flow (where i_w is the gradient of clear water having the same mean velocity as the slurry), but no detailed mechanism could be proposed. A basic difficulty in the division of the total gradient between mechanical and viscous friction causes is that there is no assurance that i_w accurately represents the part of the head losses in the slurry flow that results from viscous friction.

Newitt et al. (1955) introduced a mechanical friction coefficient in the case of heterogeneous flows, but only if a moving bed of particles existed in contact with the pipe wall. The following equation was found:

$$\frac{i - i_w}{C_d i_w} = K_3(s-1)\frac{gD}{V^2} \qquad Eq\ 5$$

where K_3 is proportional to the friction coefficient η. The drag coefficient, C_D, does not vary with size when particles are large enough (greater than about 1.5 mm in the case of sand or gravel). Then the Durand equation, with $m = 1$, takes the same form as Equation 5. With particles in that category, Newitt et al. (1955) found a good fit to this equation. Their work thus agrees with that of Babcock (1971), in finding that $m = 1$.

If a bed of particles was not allowed to form, because of lift forces of the flow, Newitt et al. (1955) apparently did not regard mechanical friction as a factor in $(i - i_w)$, even though the flow were not homogeneous. But work by Bagnold (1954, 1955, 1957) shows that the submerged weight of the particles separated from the bed can be transmitted to the bed, or directly to the pipe wall, under some conditions (see later discussion, commencing on p. 57). Mechanical friction could contribute to the head loss.

Delivered Concentration, C_d, as a Parameter

Results reported by Babcock (1971), involving only one pipe diameter, $D = 1$ inch, will be summarized here diagrammatically, in order to present compactly some of the important deviations from the Durand correlation that he described. His work also bears on

the Wilson model in a way that does not seem to have been noticed before. Figure 1(a) shows one type of plot of $(i - i_w)$ against C_d (linear scales), obtained with a series of runs, using a particular closely graded sand‡ of *intermediate* size (20/30 quartz sand — passing sieve No.20, not passing No.30 — mean size d = 0.72 mm, approximately). The points represent slurry flows at a constant velocity, V_A, at concentrations between 0.0 and 0.4.

Figure 1(b) shows another type of plot in which the plotted line is curved in the region of lower concentrations. This plot results from slurry flows with the same sand at a higher velocity, V_F.

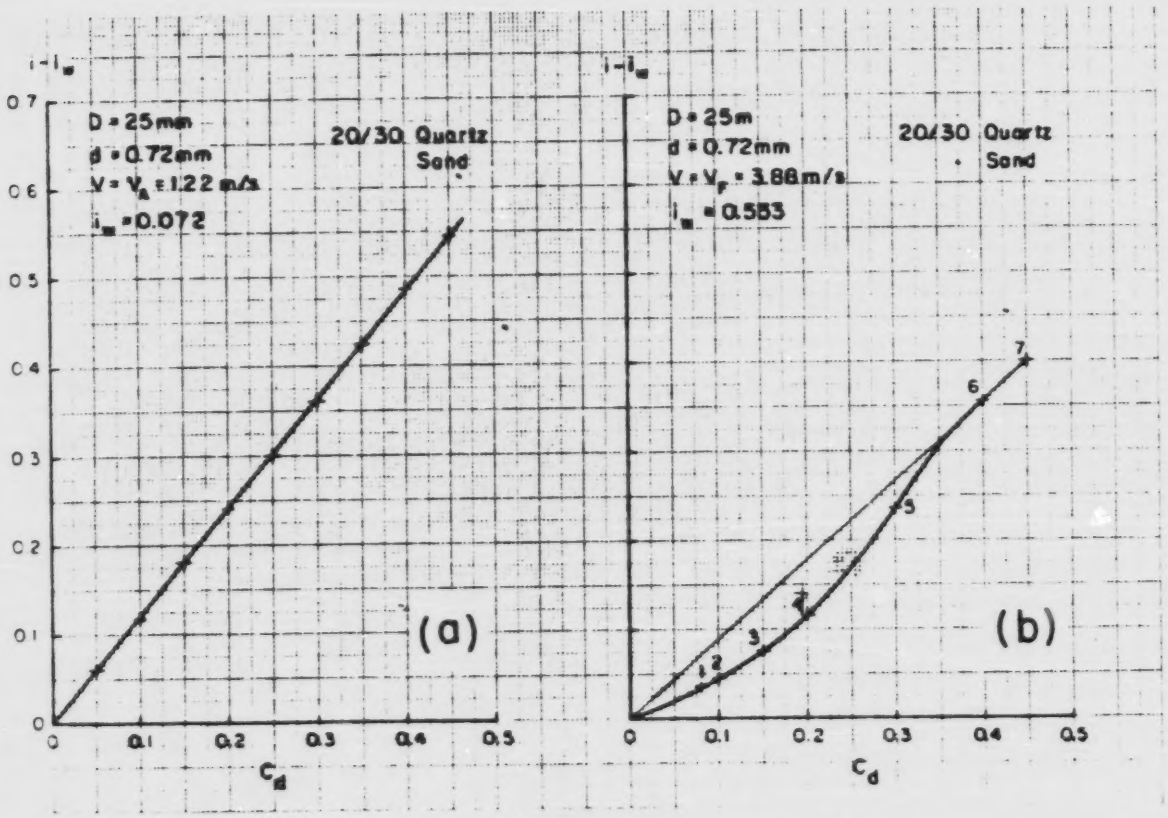

Fig. 1 — Head-loss as a function of C_d, with V constant: (a) V = 1.22 m/s; (b) V = 3.88 m/s

Figure 2 shows the resulting plot of ϕ against v (to logarithmic scales). All the points in Figure 1(a) coincide in point A in Figure 2. Similarly, points B, C, and D result

‡ This sand was called *coarse* by Babcock.

from straight line plots for velocities V_B, V_C and V_D. The straight line $A\ B\ C\ D$ satisfies the Durand equation for $m = 1$.

The points E and F, resulting from the straight line parts of plots for higher velocities, V_E and V_F, also satisfy the Durand equation. But let us now consider the points lying on the curved part of the plot for V_F in Figure 1(b). Points 5, 4, 3, 2, and 1 become F_5, F_4, F_3, F_2, and F_1 in Figure 2. Their ϕ values descend in order of the C_d values.

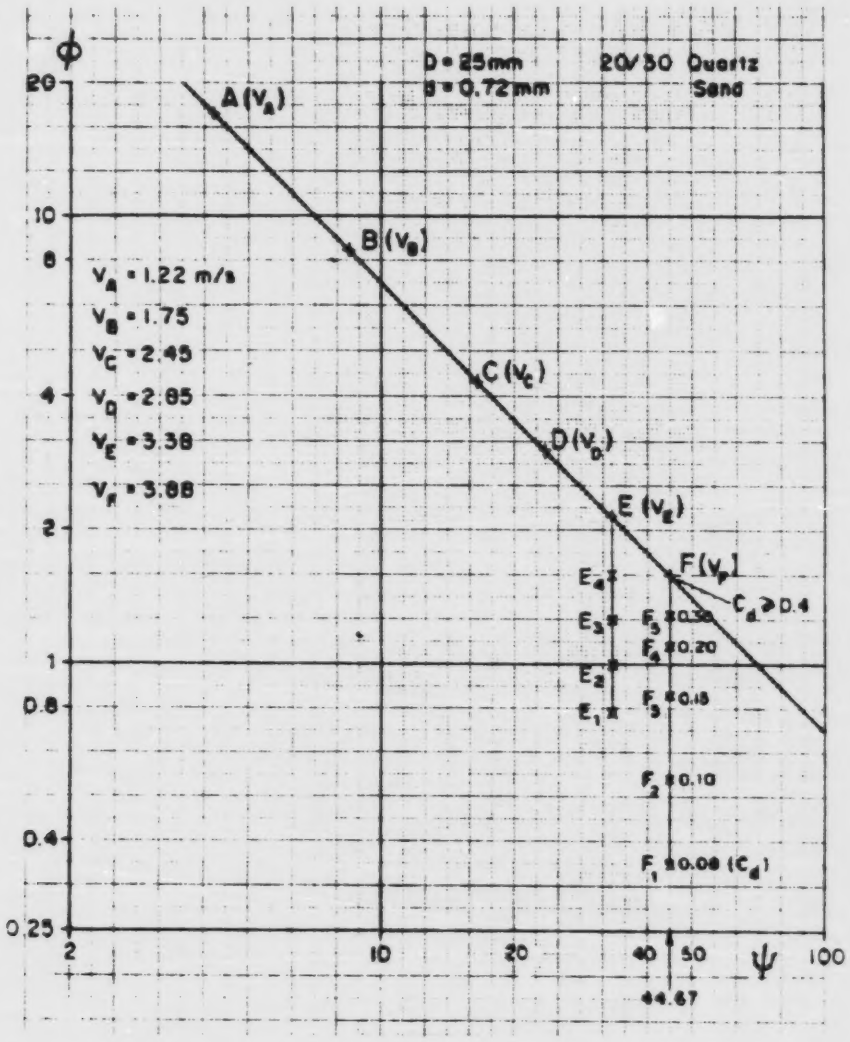

Fig. 2 — Correlation plot using the Durand equation

The same effect is shown for points E_4, E_3, E_2 and E_1. Thus, the results for lower concentrations of *intermediate* sand cannot be represented by the Durand equation at the

higher velocities†.

Similar results with an *intermediate* sand of smaller particle size (30/45 quartz sand — $d = 0.45$ mm) are shown in Figure 3. At higher concentrations the results coincide in point B', representing the same velocity V_B as in Figure 2.

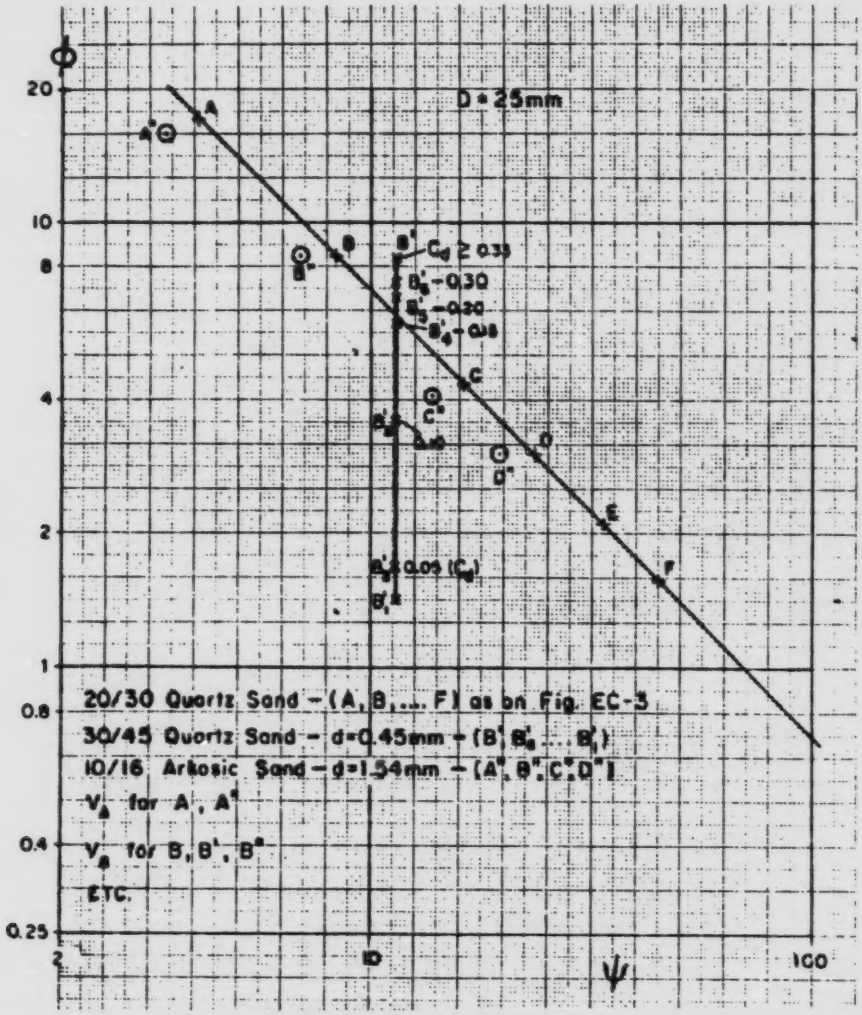

Fig. 3 — Correlation plot using the Durand equation

Points representing runs at lower concentrations, however, fall below point B', with

† Babcock did not publish a plot similar to Figure 1(b) for velocity V_E. Thus, the C_d values are not known for points E_4, E_3, E_2, and E_1. However, Figure 1(b) is given as typical and, therefore, E_1 represents a concentration of order 0.05 to 0.10, whereas E_2, E_3, and E_4 are in ascending order of the C_d values.

ϕ values decreasing in order of the C_d values. Thus, the Durand correlation fails at a lower velocity, when the size of an *intermediate* sand decreases.

Results for the 20/30 *intermediate* sand, and for a 10/16 *coarse* sand, are also plotted in Figure 3; it may be seen that the points for identical velocities do not coincide when the particle sizes are different. Babcock (1971) found, by plotting ϕ against V^2/gD instead of ψ, that the points for a given velocity did coincide. The drag coefficient, C_D, included in ψ, did not appear to compensate for particle size, as intended by Durand. In fact, C_D spoiled the correlation under the conditions described.

Babcock (1971) also tested slurries with a sand sized near the border of *fine* and *intermediate* sizes (80/100 quartz sand, with $d = 0.16$ mm). In this case he found that the results could be correlated by the Durand equation. However, he found $m = 0.25$. Thus, these results could not be well correlated with those of the *coarse* sands.

Figure 4 shows one plot of $(i - i_w)/i_w$ against C_d for this 80/100 sand, at velocity V_B. In this case the plot is linear at lower C_d values, and curves upwards at higher values.

Figure 5 shows the corresponding plot of ϕ against ψ, for velocities V_A, V_B, V_C, and V_D. The results for lower C_d values coincide in points $'A$, $'B$, $'C$, and $'D$, for velocities V_A, V_B, V_C, and V_D, respectively, as marked in Figure 2. But, it may be seen that points in the curved part of Figure 4 result in points $'B_5$, $'B_6$, and $'B_7$, in order of increasing ϕ values. This effect is opposite to that shown in Figures 2 and 3, for the *intermediate* sands.

From the point of view of a model attributing the difference $(i - i_w)$ to mechanical friction between particles and the pipe wall, the frictional force must depend on the net difference between submerged weight and hydraulic lift forces per unit volume of particles. Qualitatively, a straight plot of $(i - i_w)$ against C_d can be interpreted as indicating that the net frictional force is constant per unit volume, because of constant lift forces per unit volume.

A curved plot, such as in Figures 1(b) and 4 can be interpreted as a change in the hydraulic lift forces per unit volume with a change in C_d. For both the *intermediate* sands, and the *fine/intermediate* (80/100) sand, the net frictional force increases as C_d increases, i.e., lift forces per unit volume decrease. But for the *fine/intermediate* (80/100) sand this only occurs at higher values of C_d. For the *intermediate* sand it occurs at lower values of C_d.

For the *fine/intermediate* (80/100) sand the lift forces seem to be maximal throughout the straight section of the plot of $(i - i_w)$ against C_d, i.e., when C_d is low†. For the

† i.e., when $C_d \leq 0.23$, per Figure 4.

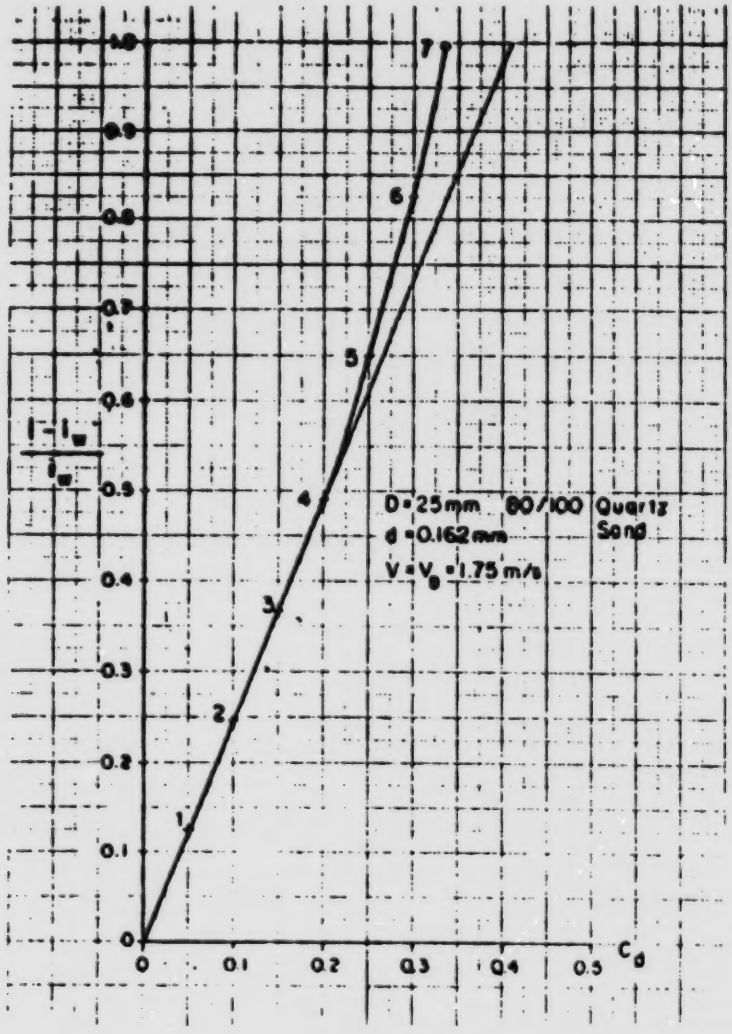

Fig. 4 — Head loss as a function of C_d, with V constant

intermediate sands the lift forces seem to be minimal throughout the straight section, i.e., when C_d is high‡. Two different flow regimes, that need not remain unchanged as the value of C_d changes, are indicated. An important consequence for the Wilson model will be discussed later.

Taken together, Figures 3 and 5 illustrate the properties of the Durand equation very well. When ψ is used on the axis of abscissae, points representing *intermediate* sands of different diameters are spread apart as mentioned previously, but points representing the *fine/intermediate* (80/100) sand line up somewhat better, very approximately, than when V^2/gD is used instead of ψ. In the latter case the point $'A$, for example, has the same value

‡ i.e., when $C_d \geq 0.4$, per Figure 1(b).

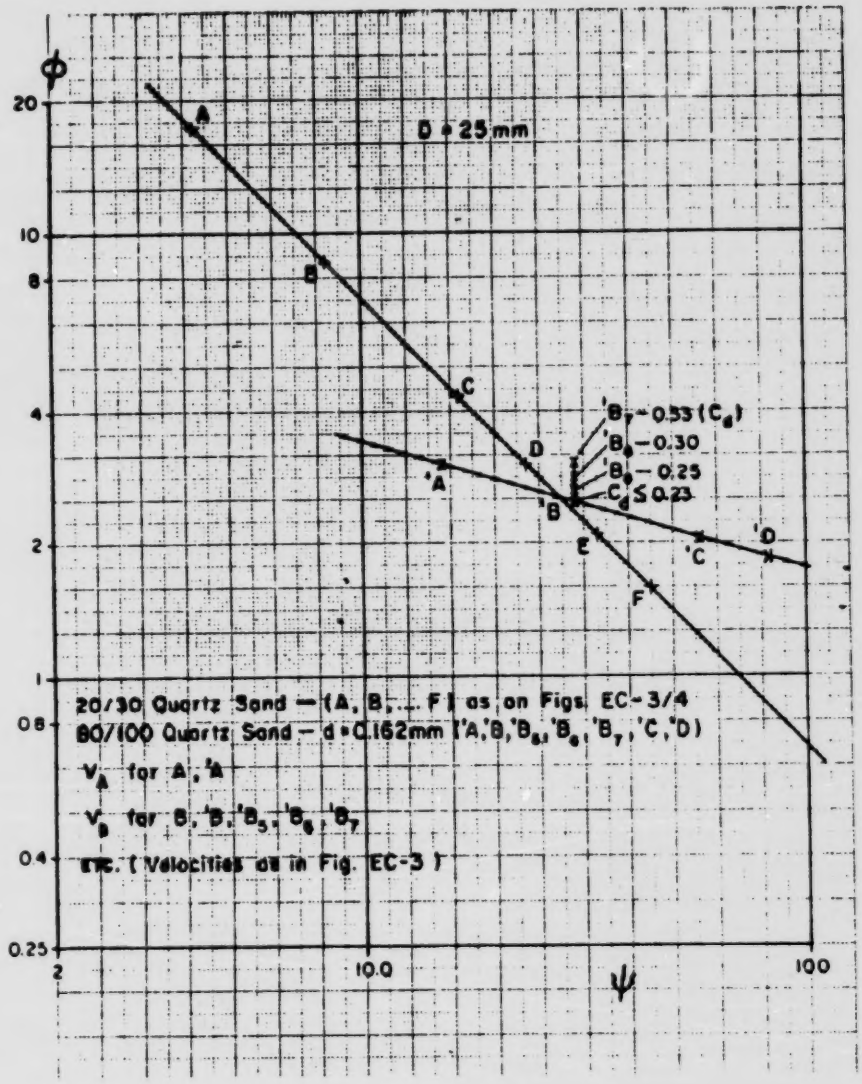

Fig. 5 — Correlation plot using the Durand equation

for V^2/gD as points A, A''. The Durand correlation can only be approximate, unless it is applied within restricted ranges of parameters, when correlations can be good.

In conclusion some remarks are perhaps also in order regarding the sometimes confusing use of the symbol ψ in the literature. As an example, it may be noted that Zandi (1971) employs the symbol $\psi = \frac{V^2\sqrt{C_D}}{gD(s-1)}$, as a dimensionless squared velocity group[§] The commonly found use of ψ agrees with Zandi (see for example Babcock, 1971). This

§ The *square root* is omitted in error in Zandi's "Notation List."

symbol is also used by the authors in this report (e.g., Equation 3).

Shook (1976) indicates agreement with Zandi, in his Equation (13). He refers to ψ in the Nomenclature List as the "dimensionless velocity"; it is, in fact, a dimensionless squared velocity. Moreover, Shook et al. (1981) also agree with Zandi in their Equation (1). Yet, in their Figures (6), (7), (8), and (9), the abscissa axis appears as ψ^{-1}, when it should actually be labelled as ψ. Similarly, Shook et al. (1982a) use ψ^{-1} on their Figures (5), (7), and (9). Again, Shook (1981) uses ψ^{-1} on his Figure (12). All these figures, as plotted, should actually have the abscissa axis labelled as ψ. Anomalous usage of ψ by Haas et al. (1980, p. 29), and by Gillies et al. (1981, p. *xiii*), may also be noted.

It appears, that no particular reason has been given for the use of ψ^{-1}. Consistent use of ψ will contribute to clarity. ψ should clearly be identified as a dimensionless squared velocity group. It is neither the reciprocal of a velocity, nor of a Froude number.

The Durand Empirical Correlation; Comments Based on Considerations of Dimensionless Analysis

It is of interest to consider the structure of the Durand equation in the light of the principles of dimensional analysis, as an application of Appendix B.

The expression $\frac{V^2}{gD(s-1)}$ can be written as:

$$\frac{V^2}{gD(s-1)} = \frac{\rho V^2}{g(\rho_s - \rho)d}\left(\frac{d}{D}\right) = \frac{\rho V^2}{\gamma_s d}\left(\frac{d}{D}\right) = \left(\overline{\overline{\Pi}}_V\right)^2 Z^{-1}$$

where, Z is the dimensionless variable D/d representing D, and $\overline{\overline{\Pi}}_V$ is the dimensionless variable representing V. These two variables belong to the complete set of dimensionless variables for representing the slurry flow, when the parameters ρ, d, and γ_s are chosen as basic parameters (see Table (B-3) and Equation (B-25) of Appendix B).

The dimensionless variable $\Xi = \frac{\gamma_s d^3}{\rho \nu^2}$ (see Table (B-3)) is also part of the system referred to ρ, d, and γ_s as basic parameters. Ξ represents the fluid viscosity μ. For a given particulate solid (with parameters γ_s, d), and a given fluid (with parameters ρ, μ), Ξ is a constant.

As shown in Appendix C, the drag coefficient C_D, for particles similar in shape, is a function of Ξ only, i.e., $C_D = \overline{\overline{\phi'}}_w(\Xi)$ (see Equation (C-4)). Thus

$$\psi = \frac{V^2\sqrt{C_D}}{gD(s-1)} = \left(\overline{\overline{\Pi}}_V\right)^2 Z^{-1}\left[\overline{\overline{\phi'}}_w(\Xi)\right]^{\frac{1}{2}} = \phi_\psi\left(\overline{\overline{\Pi}}_V,\ Z,\ \Xi\right)$$

where ϕ_ψ is a dimensionless function, that expresses ψ in terms of $\overline{\overline{\Pi}}_V$, Z, and Ξ.

In other words, ψ can be expressed in terms of valid dimensionless variables $\overline{\overline{\Pi}}_V, Z$, and Ξ, all of which are formed with the same group of basic parameters. However, the function $\overline{\overline{\phi'}}_w(\Xi)$ that is incorporated in ψ, is not a function relating to the slurry flow, but is a function relating to the terminal fall velocity of a representative particle with diameter d, in still water far from any boundary or other particles.

The right side of Equation 2, $\frac{i - i_w}{C_d i_w}$ is also anomalous, because i_w does not relate to the slurry flow; i_w relates only to a flow of clear water, having the same mean velocity as the slurry flow.

It is seen that the Durand equation unites parameters from three independent flows, or types of motion. This type of mixture of parameters is not a promising approach theoretically, but its partial empirical success has left its mark on other models.

DISCUSSION OF THE WILSON MODEL

The Wilson model was developed in stages, through a series of papers*. A detailed description of two forms of the resulting model is given in the next section. In this section, however, we discuss the nature of idealized concepts that are implied by the mathematical equations through which the model is expressed. Some inconsistencies and shortcomings of these concepts are discussed, as a contribution to a clearer understanding of the issues involved.

Shook (1983, pp. 2 and 3) refers to the difficulties in defining recognizable real phenomena characteristic of certain flow regimes. *Definitions should be restricted to recognizable phenomena, and idealizations that facilitate modelling should be clearly identified.* We attempt the second part of his recommendation. We compare some aspects of real phenomena with idealized concepts, without trying to define any real phenomenon completely.

Shook (1983, p. 2) also objects to any loose use of such terms as *sliding bed.* As the term has entered the jargon we suggest that the best way to deal with it is to adopt it. It is a useful term if it is used only to indicate the corresponding concept that is actually used in the model, as will be seen.

The Wilson model has been referred to as being the product of a *unified physically-based analysis*, and as a *mechanistic model.* These terms are acceptable, if applied only to the model, but it must be clearly understood that certain parameters and relationships of the model have not been established as representative of real hydraulic flows on a strictly physical, or mathematical, basis.

Nevertheless, workers in the field have reported that the model has been fitted successfully to some sets of experimental data. It does provide a model framework, developed by Wilson with the aid of certain general physical principles, that is an improvement on former data correlations. But the fact remains that the parameters needed to apply the model must be established through data correlation, and the limits of the range of application must be explored.

The Durand equation was also the product of the physically based thought that introduced C_D and it then became a basis for data correlation. The Wilson model is situated similarly and must be approached with an appreciation of its real nature. Some idealized concepts employed in the Wilson model are discussed in the following sub-sections.

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* Wilson (1970, 1973, 1975, 1976, May 1976, 1979, October 1982); Wilson and Brown (1982); Wilson and Judge (1977, 1978); Wilson and Watt (1975).

Sliding Bed

The sliding bed of solids upon which the Wilson model is based must consist of particles that all move together as a unit. The shear force exerted by the pipe wall on the bed is assumed to result from mechanical sliding friction between the wall and the particles that touch it. No terms are included to take account of any rotation of the particles, or any relative motion among them.

Furthermore, when the bed does not fill the pipe it is assumed to have a well-defined horizontal upper surface (in a horizontal pipe). The fluid, or slurry, flowing above the bed exerts a shear stress on the upper surface of the bed.

A flow in which the bed fills the pipe has come to be known as plug flow. At low velocity the hydraulic gradient required to maintain plug flow is represented by j_s. An important justification of the value for j_s, calculated from the Wilson model, was given in a paper by Wilson et al. (1973).

But in a closely related paper Bantin and Streat (1973) reported on the velocities of particles in a plug flow. Their Figures (8) and (9) clearly show that particles in plug flow travel at different velocities , in general, depending on their distances from the pipe wall.

Since the distortion of the bed must require energy, the calculation of j_s that forms part of the Wilson model cannot be regarded theoretically as a sound use of a normal friction factor. The agreement of the calculation with experiment should be regarded as an empirical coincidence.

Televantos et al. (1979) report on particle velocities in beds that did not fill the pipe. In their Figures (10) and (11) it can be seen that the particles in the bed did not travel at one single velocity.

The friction factor, η, as used in the model, is not a physical property that can be evaluated with confidence, independently of a flow including a bed. The applicability of an η-value (or function) must be established empirically by some process of fitting the model to sets of measurements made on the flow, as discussed later (see p. 69, l. 3, et seq.).

Eyler (1981) and Shook et al. (1982b) constructed an apparatus for measuring the sliding coefficient of friction experimentally.

Interfacial-Friction Factor, C_{f12}

The effective friction factor, C_f, of the flow above the bed of particles (assumed here to be stationary) is related, presumably, to the friction factor of the pipe wall, C_{f1}, and the

friction factor of the bed surface, C_{f12}, by the equation

$$C_f = \frac{C_{f1} S_1 + C_{f12} S_{12}}{(S_1 + S_{12})} \qquad Eq\ 6$$

where:

S_1 is the area of the pipe bounding the flow, per unit length, and

S_{12} is the area of the bed surface per unit length.

This equation rests on the following assumptions**: the cross-sectional area of the flow, A_1, is assumed to be divided into A_w, influenced by the wall only, and A_{12}, influenced by the bed only. The same pressure gradient, i, must exist in the flows through both areas, and it is assumed that the same average velocity, V, exists in both.

For flow in a clear circular pipe, we can write

$$i = 2 C_f \frac{V^2}{g D} = C_f \frac{V^2}{2 g R_h}$$

where R_h is the hydraulic radius, given by $R_h = \frac{A}{S}$, where A is the area of the flow cross section, and S is its perimeter. It is then assumed that the second expression for i can be used for a flow of any cross-sectional shape.

Thus we can write, in general,

$$i = C_f \frac{V^2 S}{2 g A}$$

or,

$$A = \frac{V^2}{2 g i} C_f S.$$

For the areas A_1, A_w, and A_{12}, we can write

$$A_1 = \frac{V^2}{2 g i} C_f (S_1 + S_{12})$$

$$A_w = \frac{V^2}{2 g i} C_{f1} S_1$$

and,

$$A_{12} = \frac{V^2}{2 g i} C_{f12} S_{12}$$

assuming that the surface that separates A_w and A_{12} does not affect either flow, because the velocities at this surface are the same for both flows.

From these equations it follows that

$$\frac{C_{f1} S_1 + C_{f12} S_{12}}{C_f (S_1 + S_{12})} = \frac{A_w + A_{12}}{A_1} = 1$$

** Wilson (1971, p. 1667); Eyler et al. (1982, Appendix F)

which reduces to Equation 6.

It appears that this relation has been used to allow for differences between C_f values assigned to different areas of a tunnel or pipe. But, if a rough bed fills a substantial part of a pipe, use of the relation is more dubious.

The assumption, that the mean velocities through A_w and A_{12} are the same, restricts the choice of the location of the hypothetical surface that separates them. The assumption, that it is neutral, is equivalent to the asumption that no net momentum, or shear stress, is transferred across it. Given the high asymmetry of the flow channel relative to its shape and surface roughness, it seems unlikely that both assumptions can be fulfilled.

Eyler et al. (1982, Section 4.1) conducted several series of experiments to measure C_f directly using beds with $\beta = 60°$, $90°$, or $120°$, and $d/D = 0.05$, 0.10, or 0.20. Knowing C_f and C_{f1} (from the known pipe roughness), they calculated C_{f12}, but they found that results for a given surface roughness were not consistent.

The values of C_{f12} corresponding to a particular Reynolds number, 2.5×10^4, with $C_{f1} = 0.0243$, were read by us from Eyler's Figure (4.7) to illustrate the nature of the results succinctly. The various values of C_{f12} are shown in Table 1.

TABLE 1
Values of the Interfacial-friction Factor, C_{f12}

d/D	C_{f12}		
	$\beta = 60°$	$\beta = 90°$	$\beta = 120°$
0.05	0.096	0.108	0.069
0.10	0.103	0.166	0.105
0.20	0.203	0.244	0.330

If Equation 6 were correct, C_{f12} values corresponding to a given value of d/D should be the same, because they are meant to characterize the surface of beds of particles of similar size, i.e., of similar roughness. The discrepancies in all three lines appear to be much greater than experimental error.

Acceptance of the rough average values of C_{f12} shown in Eyler's Tables (4.3) and (5.3), together with his formula (5.41), $\epsilon/D = 1.72\,(d/D)$, indicates a low standard of accuracy in the model, unable to match the measurement accuracy of the data on the average friction factor, C_f, of the flow.

It is unfortunate that the discussion* on the "Effect of Interfacial Shear Modelling" is not on sound ground with respect to values of C_{f12}. In this context it should also be noted, that Wilson and his coworkers have recently reexamined the flow conditions in the interfacial shear layer**.

Formula for Threshold Suspension Velocity, V_t

Wilson (1973, Figure 4) employs a plot of $\frac{i - i_w}{C_d\,(s-1)}$ against V on log scales, to determine V_t, expressing the threshold velocity for initiation of suspension due to turbulence for a particular set of pipe and particle sizes. The significance of two different contact-load relationships in which V_t (or V_*) is employed is discussed in the next subsection. Here the authors discuss Wilson's establishment of V_t.

Let X represent $\frac{i - i_w}{C_d\,(s-1)}$, for convenience.

Wilson (1973, p. 31) shows that the results of Babcock (1971) and of Newitt et al. (1955) both support the finding that X is independent of V for cases in which the immersed weight of the particles is presumed to be carried by the pipe wall, not by suspension in the fluid. Wilson proposes that if X decreases as V increases above a certain velocity V_t, this velocity marks the commencement of a partial suspension of the immersed weight of the particles by the fluid.

Wilson's (1973) Figure (4) provides one value of V_t, and he proposes that the data of Babcock (1971) provide another. Using these two values he is able to derive a preliminary relation for calculating V_t. Wilson and Watt (1975) use these two cases, combined with four further cases, to derive the final relation (their equation 6, on p. D1-7):

$$V_t = 0.6\,w\,\sqrt{\frac{8}{f'}}\,exp\,(45\,d/D)$$

where:

w = terminal fall velocity of single particle,
d = diameter of particle,
D = diameter of pipe,
f' = Darcy-Weisbach friction factor for equivalent discharge of clear water.

Returning to Wilson's (1973) use of Babcock's data, it is of interest to take a close look at Babcock's (1971) results. Wilson uses the results (see Figure 2 of this report) relating

* See Eyler et al. (1982, Section 5.7).
** Wilson (1984); Wilson and Tse (1984); Gardner (1985).

to the series of runs with 20/30 quartz sand. To facilitate a comparison with Wilson's Figure (4), the data of our Figure 2 have been replotted in terms of $X = \frac{i - i_w}{C_d\,(s-1)}$ and V, as shown in Figure 6.

The points corresponding to velocities V_A, V_B, V_C, and V_D lie on the horizontal line ab' with $X = 0.56$, referred to by Wilson as approximately 0.6. For all of these velocities X is independent of C_d.

But for higher velocities, such as V_E and V_F, X is dependent on C_d (see Figure 6). With each velocity (for example V_F), a critical value C_d' (in this case 0.4, approximately) is associated. Values of C_d less than C_d' lead to values of X less than the value 0.56 mentioned above. X decreases uniformly with C_d.

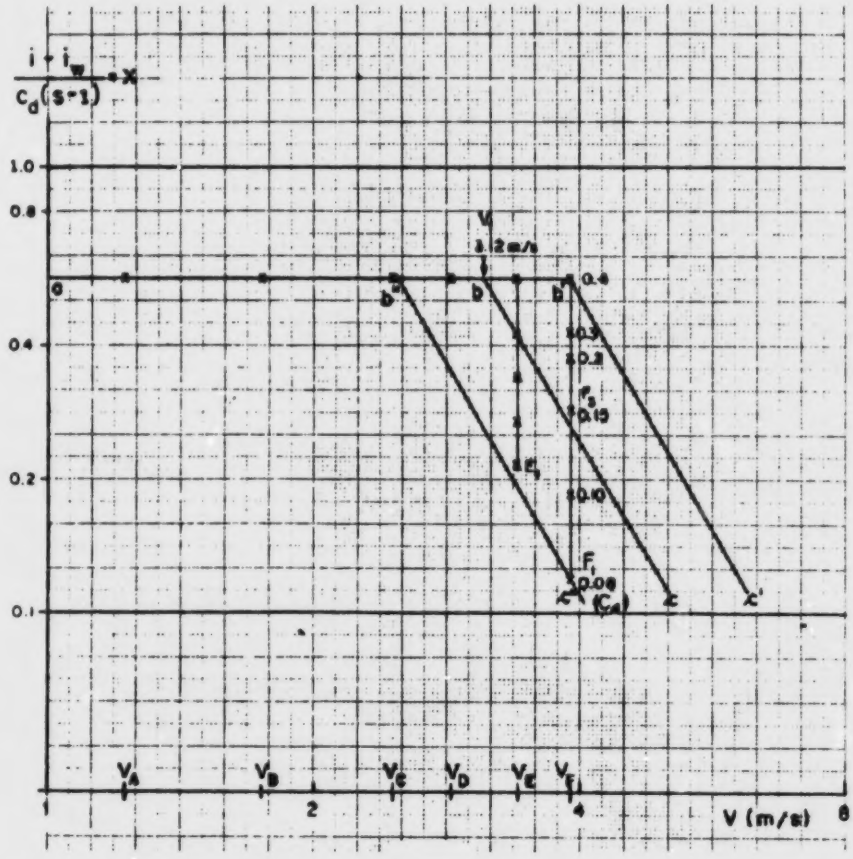


Fig. 6 — Replot of our Fig. 2 in form of Wilson's (1973) Figure (4)

In making use of Babcock's results, Wilson ignores any dependence of X on C_d. He notes that the results at V_D indicate no suspension, but that those at V_E indicate

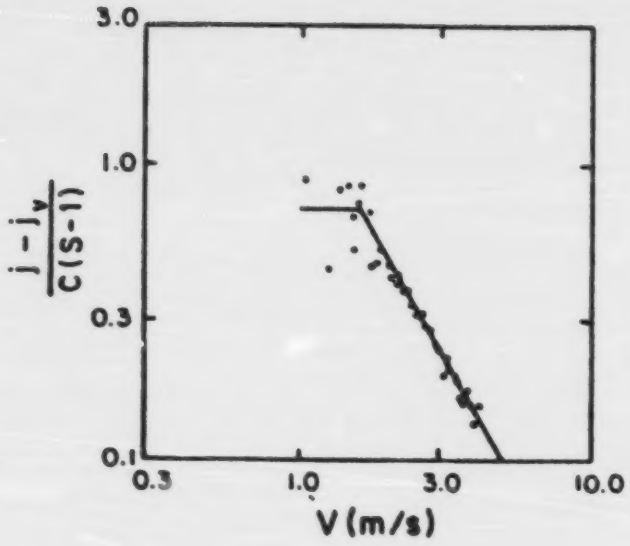


Fig. 7 — Copy of Wilson's (1973) Figure (4) — for easier reference

suspension. He therefore chooses the mean of V_D and V_E, 3.12 m/s, as V_t for this case.

On Figure 6 the position of V_t is indicated, and the point b represents the critical point in a plot similar to Figure (4) of Wilson (1973)*. A plot such as abc would be expected, where the slope of bc is chosen arbitrarily as equal to the slope (-1.7) of the right side part of Wilson's Figure (4). If V_t were independent of C_d, such a plot would be expected to represent all of the data.

But a single plot cannot represent all of Babcock's data. Each value of C_d requires a different plot. For example, $C_d = 0.4$ requires a plot such as $ab'c'$, with $V_t = V_F = 3.88\ m/s$. On the other hand, $C_d = 0.08$ requires a plot such as $ab''c''$, with $V_t \approx V_C = 2.45\ m/s$.

Clift et al. (August 1982, p. 92) state that the slope, $-M = -1.7$, has been typical of a number of tests. Consequently, the positions of the lines bc, $b'c'$, and $b''c''$ should be approximately realistic. It is, therefore, interesting to note that $b''c''$ (drawn through point F_1) passes close to E_1. The latter point represents data given by Babcock for the lowest value of C_d at velocity V_E, but this value cannot be estimated from the published data. It is likely to be close to 0.08.

We also note that bc passes close to the point F_3, representing $C_d = 0.15$ at $V = 3.88\ m/s$. This indicates that $V_t = 3.12\ m/s$ (for point b) corresponds to C_d of the order

* See Figure 7.

of 0.15, but that it is not the correct threshold suspension velocity for any value of C_d that differs appreciably from 0.15.

Because of these results it is difficult to understand the relative lack of scatter in the inclined section of the plot in Wilson's (1973) Figure (4), except by assuming that the range of C_d values is rather small.

In the discussion relating to their formula for V_t, Wilson and Watt (1975) make no mention of the values of C_d in the data used. The formula must represent a sort of mean value for V_t, but a certain distortion could have entered into it, if the sets of data used to develop it correspond to different mean values of C_d.

Eyler and Lombardo (1980, p. 3.9) note that the Wilson and Watt (1975) formula for V_t predicts an unreasonably high value, giving one particular example with $\overline{d}/D \approx 0.07$. They remark that the formula is based on data for which $d/D \leq 0.03$. Use of the formula with d/D greater than this limit may be one factor in a questionable result. But the neglect of C_d in establishing, and in using, the formula may also be an important factor.

Shook et al. (1981, pp. 88 and 89) report satisfactory predictions by the Wilson model with high concentrations of solids, but less satisfactory results with low concentrations. Aside from any other sources of discrepancy, the V_t formula may be a factor.

It is noteworthy that no development has taken place in the V_t formula since Wilson and Watt (1975). Let us write the formula in the general form

$$V_t = A\, w \sqrt{\frac{8}{f'}}\, exp\,(B\, d/D). \qquad Eq\ 7$$

In view of Babcock's work it would be desirable to determine A and B as functions of C_d (or C_t if the model uses this). A large amount of experimental work would be required to cover a wide range of C_d values, and to extend the d/D range above 0.03.

In principle, however, this is a relatively simple way of making a possible improvement to the Wilson model.

Contact-load to Total-load Relationships

In the literature the following equations can be found to express the above relationships:

$$\frac{C_c}{C} = \left(\frac{V_t}{V}\right)^{\alpha} \qquad \text{(see Wilson May 1976, Eq 2)}$$

and,

$$R = \left(\frac{V_u}{V_m}\right)^M \qquad \text{(see Clift et al. Aug. 1982, Eq 3)}$$

Wilson (May 1976) states that his equation

$$\frac{C_c}{C} = \left(\frac{V_t}{V}\right)^\alpha$$

would appear to represent approximately the relation between contact load and total load of particles. In Wilson's notation C_c and C represent contact and total volumetric loads, respectively, as delivered. Also,

V_t = threshold velocity for initiation of turbulent suspension; and

V = slurry throughput velocity.

In the notation used in this report, the relation becomes

$$\frac{C_{dc}}{C_d} = \left(\frac{V_t}{V}\right)^\alpha. \qquad Eq\ 8$$

Clift et al. (August 1982) introduce the *stratification ratio, R,* — defined to be *the fraction of the total conveyed solids moving as non-suspended load.* In the context of the paper it is clear that the total conveyed solids must be regarded as delivered load, C_d, and the non-suspended load (contact load) as a fraction of delivered load, i.e., C_{dc}. Therefore,

$$R = \frac{C_{dc}}{C_d}. \qquad Eq\ 9$$

Clift et al. (August 1982) write

$$R = \left(\frac{V_u}{V_m}\right)^M.$$

In the notation of this report this becomes

$$R = \left(\frac{V_t}{V}\right)^M \qquad Eq\ 10$$

and it follows that

$$\frac{C_{dc}}{C_d} = \left(\frac{V_t}{V}\right)^M. \qquad Eq\ 11$$

On comparing Equations 8 and 11 we would expect α and M to be identical. However, Wilson (May 1976, p. 4) states, without explanation, that *the exponent α has a value slightly less than 2.* In fact, $\alpha = 2$ is commonly assumed in the literature concerned with the Wilson model.

On the other hand, Clift et al. (Aug. 1982, p. 92) report *typical test results giving a value of M close to 1.7.* The authors (in their notation) determine M as the absolute value of the slope of a plot of $\frac{i_m - i_f}{s_m - 1}$ against V_m, on logarithmic scales. Figures (4), (5), and (7) of their paper, in fact, show M values of 1.54, 1.68, and 1.68, respectively.

Figure (4) of Wilson (1973) also shows a similar plot with an M-value of 1.73. Wilson (May 1976) makes reference to this work, but clearly he did not determine α directly from plots like those in the four figures mentioned above. He, therefore, must have determined α in some manner different from that given by Clift et al. (Aug. 1982). It will be shown here that an explanation for the relation between α and M can be drawn from a consideration of the two papers — namely of Wilson (1973), and of Clift et al. (Aug. 1982) — taken together.

Equation (15), of Clift et al. (August 1982), states that

$$\frac{(i_m - i_f)}{(s_m - 1)} = A' i_f \left[1 - \left(\frac{V_u}{V_m}\right)^M\right] + B \left(\frac{V_u}{V_m}\right)^M$$

where*, $(s_m - 1) = (s_s - 1) C_{vd}$.

In the notation of our report this becomes

$$\frac{(i - i_w)}{(s - 1) C_d} = A' i_w \left[1 - \left(\frac{V_t}{V}\right)^M\right] + B \left(\frac{V_t}{V}\right)^M \qquad Eq\ 12$$

where, $s = \rho_s / \rho_w$.

This equation is assumed to hold only for $V \geq V_t$, and it is stated that *whenever a significant part of the solids is moving as a stratified load, the term in B dominates the equation*‡. The text indicates that under these conditions the term in their Equation (5) that contains A' can be ignored, and that M can be determined by plotting $X = \frac{(i - i_w)}{(s - 1) C_d}$ against V on logarithmic scales.

The plot is then interpreted in terms of the reduced form of Equation (5), i.e.,

$$\frac{(i - i_w)}{(s - 1) C_d} = B \left(\frac{V_t}{V}\right)^M \qquad Eq\ 13$$

which is true only for $V \geq V_t$.

Figure 8 gives an example of such a plot, represented by the line tm, part of the line Ltm. Point L corresponds to V_L, the *critical deposit velocity.* At velocities less than V_L a

* Per Equation (5) of Clift et al. (August 1982).

‡ *Stratified load* means contact load.

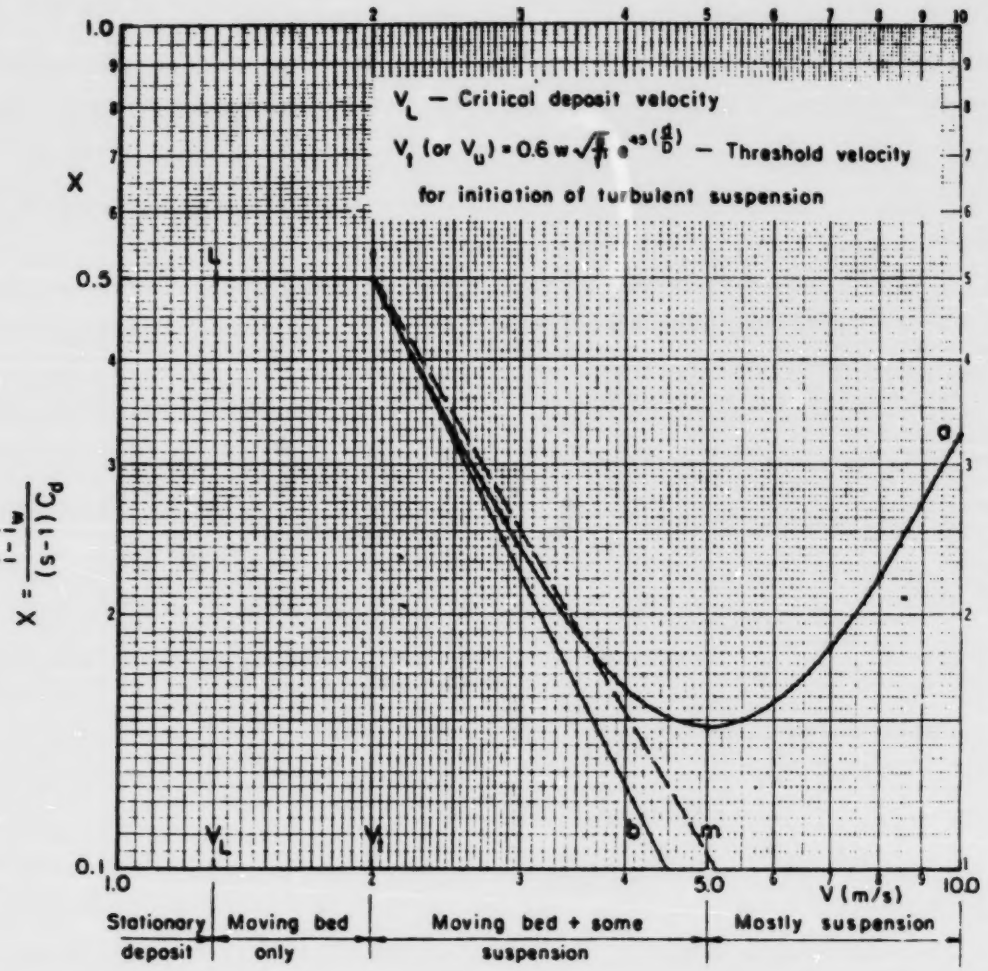


Fig. 8 — Plot, as per Equation 13

stationary deposit of particles exists at the bottom of the pipe†. Concepts treated in the paper by Clift et al. (August 1982) do not apply at velocities less than V_L.

Point t corresponds to V_t, the velocity at which the fluid begins to exert a dynamic support force on the particles. At velocities between V_L and V_t the particles in the flow all travel as contact load, and the value of X is B. At increasing velocities (greater than V_t) dynamic support acting on the particles causes X to fall progressively below B.

† Normally a minimal quantity of stationary particles, that can be observed, is required to define V_L; in the literature a number of expressions can be found for the *critical deposit velocity*, such as those by Durand-Condolios, and by Zandi-Govatos.

B and V_t are determined by the observations mentioned above. M is determined by the slope of the mean straight line tm that best fits the data.

If the size, shape, and density of the particles are such that the particles cannot travel totally as contact load at any observed velocity, no horizontal section Lt of the line Ltm can be observed. Point t then becomes simply the upper end of the mean straight line through the plotted points.

Under these circumstances, B and V_t cannot be defined. The product $B\,V_t^M$ in Equation 13 must be replaced by K, a constant, to give

$$\frac{(i - i_w)}{(s-1)\,C_d} = \frac{K}{V^M} \qquad Eq\ 14$$

for which K and M can be determined from the line tm.

Now let us substitute α for M in Equation 12 to obtain

$$X = \frac{(i - i_w)}{(s-1)\,C_d} = A'\,i_w \left[1 - \left(\frac{V_t}{V}\right)^{\alpha}\right] + B\left(\frac{V_t}{V}\right)^{\alpha} \qquad Eq\ 15$$

The line Lta in Figure 8 is a plot of $X = \frac{(i - i_w)}{(s-1)\,C_d}$ against V, based on the complete Equation 15. The assumptions used in calculating points on the line are given in Table 2, together with data for plotting the points. This line is similar to that shown in Figure (2) of Clift et al. (August 1982).

The assumptions are intended to provide a basis for simulating approximately some of the data plotted in Figures (6) and (7) of Clift et al. (August 1982), in particular the data taken with $C = 0.11$ and $D = 0.20\ m$, and plotted with open triangles.

The assumption, $\alpha = 2.0$, is made in accordance with Wilson's practice.

The assumptions, $A' = 1.0$ and $f_w = 0.0125$, relate to the use of Equation (12) of Clift et al. (August 1982), which can be written (with M replaced by α), as

$$\frac{(f - f_w)}{(s-1)\,C_d} = A'\,f_w \left[1 - \left(\frac{V_t}{V}\right)^{\alpha}\right] + B\,\frac{2\,g\,D}{V^2}\left(\frac{V_t}{V}\right)^{\alpha} \qquad Eq\ 16$$

where, $i = \frac{f\,V^2}{2\,g\,D}$, and $i_w = \frac{f_w\,V^2}{2\,g\,D}$. In these equations f and f_w are Darcy-Weisbach friction factors for the mixture and for clear water, respectively.

A plot of $Y = \frac{(f - f_w)}{(s-1)\,C_d}$ against V on linear scales is shown in Figure (1) of Clift et al. (August 1982). As V becomes large, Y approaches $A'\,f_w$, and Equation 16 can be approximated by

$$\frac{(f - f_w)}{(s-1)\,C_d} = A'\,f_w. \qquad Eq\ 17$$

TABLE 2

Data for Line *Lta* in Figure 8

$V \leq V_t$: $X = 0.500$

$V \geq V_t$: $X = A' i_w \left[1 - \left(\frac{V_t}{V}\right)^{\alpha}\right] + B\left(\frac{V_t}{V}\right)^{\alpha}$

$A' = 1.0$, $B = 0.5$, $\alpha = 2.0$, $V_t = 2.0\ m/s$, $i_w = \frac{f_w V^2}{2 g D}$,
$D = 0.2\ m$, $g = 9.81\ m/s^2$, and $f_w = 0.0125$.

V	$A' i_w \left[1 - (V_t/V)^2\right]$	$B (V_t/V)^2$	X
2.000	0.0000	0.5000	0.500
2.222	0.0030	0.4050	0.408
2.500	0.0072	0.3200	0.327
2.857	0.0133	0.2450	0.258
3.333	0.0226	0.1800	0.203
4.000	0.0382	0.1250	0.163
4.500	0.0518	0.0988	0.151
5.000	0.0669	0.0800	0.147
5.500	0.0836	0.0661	0.150
6.000	0.1020	0.0556	0.158
7.000	0.1434	0.0408	0.184
8.000	0.1912	0.0312	0.222
10.000	0.3058	0.0200	0.326
20.000	1.2617	0.0050	1.267
40.000	5.0848	0.0012	5.086
100.000	31.8473	0.0002	31.847

NOTE: f_w varies with V. Estimates show that this variation has a negligible effect on the curve *Lta* with respect to the matter discussed.

In Figure (6) of Clift et al. (August 1982) a horizontal line is shown as the estimated limit value, $A' f_w$, of the data shown with open triangles. The line is marked f_f (f_w in

present notation), indicating that $A' = 1$, as mentioned in their text (see p. 97). The position of the line f_f on the plot shows that $f_f = f_w = 0.0125$.

The assumption, $B = 0.5$, is made so that the minimum in the curve *Lta* will occur in the neighbourhood of $V = 5\ m/s$, because the open triangles in Figure (7) of Clift et al. (August 1982) depart from the mean straight line near that value of V. A minimum consistent with the data should occur at a velocity not much greater than $5\ m/s$. Clift et al. (August 1982) say (see p. 93) that "Newitt et al. (1955) found $B = 0.8$, but lower values have often been found."

It is finally assumed that $V_t = 2.0\ m/s$, because $X = B = 0.5$ occurs at $V = V_t = 2.0$ on the mean line of Figure (7) of Clift et al. (August 1982) The point $(B,\ V_t)$ so defined is not inconsistent with the open-triangle points.

If Equation 15 represents concepts that are useful, the line *Lta* should simulate the mean line through experimental data from an experiment in which the assumptions of Table 2 were fulfilled. Because of experimental scatter, a mean straight line, similar to *tm*, would be drawn, with a slope of about $-M = -1.7$.

But if Equation 15 were reduced to

$$\frac{(i - i_w)}{(s-1)\,C_d} = B\left(\frac{V_t}{V}\right)^{\alpha} \qquad Eq\ 18$$

as Equation 12 was reduced to Equation 13, the line *Ltb* would be expected as an approximation, with the section *tb* having the slope $-\alpha = -2.0$.

Thus *tb* is not a linear approximation to the data, whereas *tm* is. Conversely, the full Equation 12 with $M = 1.7$ does not represent the data, whereas Equation 15 does.

M and α have distinctly different rôles. M is associated with a straight line interpretation of data (Equation 13), whereas α is associated with a more complicated interpretation (Equation 15).

It seems likely that Wilson (May 1976) had an equation such as Equation 15 in mind when he stated that $\alpha = 2.0$ (approximately), knowing that plots such as that of Figure (4) in Wilson (1973) showed slopes in the neighbourhood of $-M = -1.7$. However, the connection between α and M has not been published, to the knowledge of the present writers. Clift et al. (1982) give the basis for Equation 15 but their derivation of the approximate equation is incorrect and misleading.

It can be seen that, in relation to given data, M must be less than α, and its value will depend on the range of velocities treated. A theoretical basis for α has not been given, nor has a reason been stated for assuming that its value is a constant, and equal to 2.

COMPARATIVE CALCULATIONS RELATING TO A SIMPLE WILSON MODEL

In this section the authors make use of the results of an illustrative example of an application of the Wilson model (see Wilson, May 1976, p. 8), to compare these results with those that would follow if the equations given by Shook (1981, pp. 16 to 19) were applied to the same basic flow data.

It will be seen that Wilson's Figure (3) (May 1976, p. A1-14), basic to the illustrative example, provides results that differ greatly from calculations given here. Unfortunately, Wilson did not give equations to handle cases with a moving bed, or with a part of the solids in suspension. Therefore, it has been impossible to discover a reason for the discrepancy. However, it is useful to use Wilson's example as a basis for a discussion of the equations given by Shook.

It may be mentioned that the equations given for a sliding bed by Eyler et al. (1982, p. A.15) as an extension of Wilson's fixed-bed equations, also do not treat cases with suspended particles in the slurry.

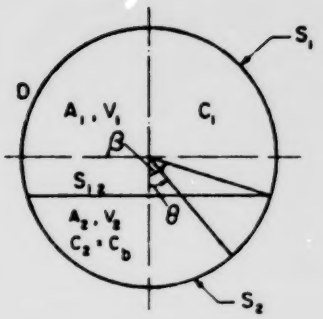
In carrying out the comparative calculations here, the authors intend to incorporate the main idea behind Shook's equations, as well as those advanced by Eyler et al. (1980, 1982). However, only the simplest model, of the type discussed by the latter authors, is adopted. It is intended that the parameters chosen should agree as closely as possible with the estimates made by Wilson. The model used by Wilson will be called "Model A," and the one by Shook, "Model B."

The basic difference between the models is that Model A uses the two force balance equations (A3) and (A4) (see Shook 1981, p. 17) to derive only the additional head loss due to contact-load, whereas Model B uses these equations to calculate the total head loss of the flow. This approach was introduced by Televantos et al. (1979). The full set of equations can not be solved; Shook (1981) and Shook et al. (1981) give a simplified version that can be solved by an iterative process (see Shook, 1983, p. 15, and Eyler et al., 1982, Appendix A).

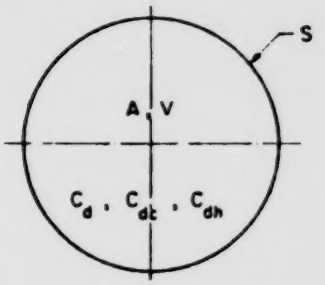
Basic Assumptions

Figure 9 summarizes the main parameters of the Wilson model.

A flow in a pipe of diameter, D, consists of a compact bed of particles with interstitial fluid moving through area A_2, and a slurry of suspended particles moving through area A_1. S_{12} is the area of the interface between bed and slurry, per unit length of pipe. Similarly, S_1



(a) Parameters in situ



(b) Parameters as delivered

Fig. 9 — The parameters of the Wilson model

and S_2 are the wall areas, per unit length, bounding the slurry and the bed, respectively. Angle β, measured in the sense shown, specifies the level of the top of the bed. Angle θ similarly specifies a variable level in the bed. C_1 is the in situ concentration of the solids suspended in the slurry. C_1 applies only to area A_1 (see *Discussion* on p. 67). All concentrations are stated in volumetric terms.

$C_2 = C_b$ is the constant in situ concentation of the solids in the bed. The submerged weight of these solids is assumed to be supported by the pipe wall. No dynamic hydraulic support is provided by the fluid. Note that these assumptions, regarding C_1 and C_2, differ from those of Shook (1981), represented by his equation $C_2 = C_{max} - C_1$ [A16]. His equation numbering is given here in square brackets.

The average velocity of the slurry is V_1, which applies both to particles and fluid. The velocity of the bed is V_2; the fluid and all particles in the bed have the same velocity. The foregoing in situ parameters are shown in Figure 9(a).

Figure 9(b) illustrates the delivered parameters, as if a fully mixed flow occurred in the pipe. The total cross-section is $A = A_1 + A_2$. V is the average velocity of the flow. C_d is the total delivered concentration of the solids (total delivered load). C_{dc} is the delivered concentration of the solids that travelled in the bed (delivered contact load). C_{dh} is the delivered concentration of solids that travelled in the slurry (delivered suspended load; subscript h indicates homogeneous flow in the slurry, in situ).

The in situ density of the slurry in A_1 is

$$\rho_1 = \rho_w[1 + (s-1)C_1] \qquad [A6] \quad Eq\ 19$$

where:

ρ_w is the density of water, and
$s = \rho_s/\rho_w$, where, ρ_s is the density of the solids when saturated with water (i.e., the "effective" density).

The in situ density of the bed in A_2 is

$$\rho_2 = \rho_w[1 + (s-1)C_b]. \qquad Eq\ 20$$

The density of the delivered mixture is

$$\rho_d = \rho_w[1 + (s-1)C_d]. \qquad Eq\ 21$$

The equations given by Shook (1981) are adapted to Models A (Wilson), and B (Shook), and to the notation of this report, as follows:

$$AV = A_1V_1 + A_2V_2 \qquad [A1] \quad Eq\ 22$$

$$C_dAV = C_1A_1V_1 + C_bA_2V_2 \qquad [A2] \quad Eq\ 23$$

$$i_1\rho_w g = \frac{\tau_1 S_1 + \tau_{12}S_{12}}{A_1} \qquad [A3] \quad Eq\ 24$$

$$i_2\rho_w g = \frac{\overline{\tau}_2 S_2 - \tau_{12}S_{12}}{A_2}. \qquad [A4] \quad Eq\ 25$$

When a solution is obtained, the head losses (in metres of clear fluid per metre of pipe) are all equal, i.e., $i_1 = i_2 = i$. As for the other symbols in the foregoing equations:

τ_1, τ_{12}, and $\overline{\tau}_2$ are the shear stresses across the areas S_1, S_{12}, and S_2, with

$$\tau_1 = C_{f1}V_1^2\rho_1/2 \qquad [A5] \quad Eq\ 26$$

$$\tau_{12} = C_{f12}(V_1 - V_2)^2\rho_1/2 \qquad [A10] \quad Eq\ 27$$

$$\overline{\tau}_2 = \tau_{2f} + \tau_{2\eta} \qquad [A14] \quad Eq\ 28$$

where:

$$\tau_{2f} = C_{f1} V_2^2 \rho_1 / 2 \qquad [A15] \quad Eq\ 29$$

$$\tau_{2\eta} = \eta (D/S_2) \int_0^{\beta} \sigma_{r*} d\theta, \qquad Eq\ 30$$

and,

$$\sigma_{r*} = (\rho_* - \rho_w) C_b (D/2)(cos\theta - cos\beta)\, g. \qquad [A13] \quad Eq\ 31$$

C_{f1}, and C_{f12} are Fanning friction factors.

σ_{r*} is the pressure, at depth $y = (D/2)(cos\theta - cos\beta)$ below surface S_{12}, due to the submerged weight of the contact solids. This pressure is assumed to act normally on surface S_2, as a liquid pressure would. The term $\tau_{12}/tan\phi$ is omitted from Equation 31, because, as is discussed later (see p. 57) its inclusion is shown to be untenable.

The equation derived by Wilson for calculating C_{dc}, is written for each size fraction as

$$(C_{dci}/C_{di}) = 0.36\, (w_i/u_*)^2\, exp\,(90 d_i/D) \qquad [A17] \quad Eq\ 32$$

where:

C_{di}	is the total delivered concentration of particles in the *ith* size fraction,
C_{dci}	is the part of C_{di} that is in the contact-load,
w_i	is the mean terminal fall velocity for the particles,
d_i	is the mean particle diameter, and
u_*	$= (igD/4)^{\frac{1}{2}}$, is the friction velocity of the flow.

Then,

$$C_{dc} = \sum_{i=1}^{N} C_{dci}$$

where, N is the number of size fractions.

It should be noted that the constants, 0.36 and 90, in our Equation 32, follow from Equations (1) and (2) of Wilson (1976). All concentrations used in the equations, and in the experimental results on which the constants were evaluated, were expressed as delivered concentrations. Although it would be more satisfactory, on general grounds, to express our Equation 32 in terms of in situ concentrations, the authors are not aware of any logical justification in the referenced literature for simply substituting in situ concentrations in the same relation. Shook's equation [A17] in terms of in situ concentrations can only be looked upon as a speculative attempt to find a new empirical relation that gives better results than Wilson's original relation (which, of course, is also largely empirical).

Relations Useful for Solving the Model Equations

$$S = S_1 + S_2 = \pi D \qquad Eq\ 33$$

$$S_2 = \beta D \qquad Eq\ 34$$

$$S_1 = (\pi - \beta)D \qquad Eq\ 35$$

$$S_{12} = D \sin\beta \qquad Eq\ 36$$

$$A = A_1 + A_2 = \pi D^2/4 \qquad Eq\ 37$$

$$A_2 = (D^2/4)(\beta - \sin\beta\cos\beta) \qquad Eq\ 38$$

$$A_1 = (D^2/4)[\pi - (\beta - \sin\beta\cos\beta)] \qquad Eq\ 39$$

$$\int_0^{\beta} (\cos\theta - \cos\beta)d\theta = \int_0^{\beta} d(\sin\theta - \theta\cos\beta) = (\sin\beta - \beta\cos\beta). \qquad Eq\ 40$$

Because C_b represents contact-load particles in situ (i.e., in the bed), and C_{dc} represents contact-load particles as delivered, $C_b A_2 V_2 = C_{dc} A V$. Therefore,

$$V_2 = \frac{C_{dc}}{C_b}\frac{V}{(1-a)} \qquad Eq\ 41$$

where: $a = \frac{A_1}{A}$, and $\frac{A_2}{A} = \frac{A - A_1}{A} = (1 - a)$.

From Equation 22,

$$V_1 = \frac{V - (1-a)V_2}{a} = \frac{V}{a} - \frac{(1-a)}{a}\frac{C_{dc}}{C_b}\frac{V}{(1-a)}$$

therefore,

$$V_1 = \left(1 - \frac{C_{dc}}{C_b}\right)\frac{V}{a}. \qquad Eq\ 42$$

From Equation 23, $\qquad C_1 = \frac{C_d V - C_b(1-a)V_2}{aV_1}.$

By substituting for V_1 and V_2, from Equations 41 and 42, we find that

$$C_1 = C_d\frac{(1 - C_{dc}/C_d)}{(1 - C_{dc}/C_b)}. \qquad Eq\ 43$$

Equations 24 and 25 may be written as follows:

$$i_1 g = \frac{\tau_1}{\rho_w}\frac{S_1}{A_1} + \frac{\tau_{12}}{\rho_w}\frac{S_{12}}{A_1} \qquad Eq\ 44$$

and,

$$i_2 g = \frac{\overline{\tau}_2}{\rho_w}\frac{S_2}{A_2} - \frac{\tau_{12}}{\rho_w}\frac{S_{12}}{A_2}. \qquad Eq\ 45$$

The following relations are derived from Equations 34 to 39:

$$\frac{S_1}{A_1} = \frac{4}{D}\frac{\pi - \beta}{[\pi - (\beta - \sin\beta\cos\beta)]} \qquad Eq\ 46$$

$$\frac{S_2}{A_2} = \frac{4}{D}\frac{\beta}{(\beta - \sin\beta\cos\beta)} \qquad Eq\ 47$$

$$\frac{S_{12}}{A_1} = \frac{S_1}{A_1}\frac{\sin\beta}{(\pi - \beta)} \qquad Eq\ 48$$

$$\frac{S_{12}}{A_2} = \frac{S_2}{A_2}\frac{\sin\beta}{\beta} \qquad Eq\ 49$$

$$\frac{S_2}{A} = \frac{\beta D}{D^2\frac{\pi}{4}} = \frac{4}{D}\frac{\beta}{\pi} \qquad Eq\ 50$$

and,

$$(1 - a) = \frac{A_2}{A} = \frac{\beta - \sin\beta\cos\beta}{\pi}. \qquad Eq\ 51$$

From Equations 30, 40, and 50,

$$\begin{aligned}\frac{\tau_{2\eta}}{\rho_w} &= \eta\frac{D}{S_2}\frac{1}{\rho_w}\int_0^\beta \sigma_{r_s} d\theta \\ &= \eta\frac{D^2}{2S_2} C_b (s-1) g \int_0^\beta (\cos\theta - \cos\beta) d\theta \\ &= 2\eta\frac{\pi D^2/4}{S_2}(s-1) g C_b \frac{(\sin\beta - \beta\cos\beta)}{\pi} \\ &= g j_p \left(\frac{S_2}{A}\right)^{-1} \phi_1(\beta) \qquad Eq\ 52\end{aligned}$$

where,

$$j_p = 2\eta(s-1) C_b \qquad Eq\ 53$$

and,

$$\phi_1(\beta) = \frac{(\sin\beta - \beta\cos\beta)}{\pi}. \qquad Eq\ 54$$

Parameters for Wilson's Model A and Shook's Model B

The basic parameters given by Wilson (May 1976, pp. 8 and 9), are as follows:

$D = 0.4\,\text{m}$; $s = 2.65$; $\eta = 0.40$; $C_d = 0.18$; $C_b = 0.6.$; $g = 9.81\,\text{m/sec}^2$;

and $\zeta = 4.4$ (see Wilson 1976, p. A1 – 9).

The following assumptions, made by Wilson, have been adopted. Wilson stated, that $f' = f_0 = 0.013$ was a suitable Darcy-Weisbach friction factor for the flow of clear water in a $D = 0.4$ m diameter pipe of commercial roughness, in the velocity range considered. The equivalent Fanning friction factor is $C_{f0} = f_0/4 = 3.25 \times 10^{-3}$.

Wilson also assumed that, in Equations 26 and 29, $C_{f1} = C_{f0}$. This is an empirical assumption, dating back at least to Newitt et al. (1955, Equation 7), that is commonly made for slurry flow. It replaces the following equations given by Shook:

$$C_{f1} = C_{f1}[D_{el} V_1 \rho_1/\mu_1,\ \epsilon/D_{el}] \qquad [A6]$$

or,

$$C_{f1} = C_{f1}[D V \rho_1/\mu_1,\ \epsilon/D] \qquad [A7]$$

where,

$$D_{el} = 4 A_1/(S_1 + S_{12}) \qquad [A8]$$

and,

$$\frac{\mu_1}{\mu_f} = \exp\left[\frac{C_1}{(0.2692 - 0.2254 C_1)}\right]. \qquad [A9]$$

The foregoing equations represent a more sophisticated approach to the problem, but all variants of the Wilson model are, by sheer necessity, so simplified theoretically, that hope for improvement is speculative. The added complexity tends to obscure the properties of the basic model, the discussion of which is the purpose of the present authors.

Wilson also assumed that $C_{f12} = \zeta C_{f0}$, where ζ is a function depending on the characteristics of the flow, including wall roughness, sizes of particles in the bed, and in the slurry, as well as the shape of the cross-section A_1. The equation

$$C_{f12} = \frac{2}{[4 \log (D_{el}/d_m) + 2.28]^2} \qquad [A11]^{\dagger} \quad Eq\ 55$$

is used to calculate ζ. On the basis of his calulations, Wilson simply assumed that, in relation to his Model A, $\zeta = 4.4$. Its use in Shook's Model B is the same.

Thus, using Equations 19, 26, 27, and 29, the shear stresses (in units of (N/m^2), are:

$$\tau_1 = C_{f0} V_1^2 \rho_1/2 = C_{f0} V_1^2 \rho_w[1 + (s-1)C_1]/2 \qquad Eq\ 56^{\ddagger}$$

$$\tau_{12} = \zeta C_{f0} (V_1 - V_2)^2 \rho_w [1 + (s-1) C_1]/2 \qquad Eq\ 57^{\S}$$

† See also Eq 75. The value 3.36 in Shook's Eq [A11] is replaced by 2.28.
‡ Since $C_{f1} = C_{f0}$.
§ Since $C_{f12} = \zeta C_{f0}$.

$$\tau_{2f} = C_{f0}\, V_2^2\, \rho_w\, [1 + (s-1)\, C_1]/2 \qquad Eq\ 58$$

with,

$$C_{f0} = 3.25 \times 10^{-3}$$

and,

$$\zeta = 4.4.$$

Wilson's Model A requires an additional equation, appropriate to the flow of the slurry, defined for A_1, when it is assumed to be flowing at velocity V in the full area A of the pipe, in the absence of contact load. The shear stress is calculated as

$$\tau_0 + \tau_h = C_{f0}\, V^2\, \rho_{dh}\, /2 = C_{f0}\, V^2\, \rho_w\, [1 + (s-1)\, C_{dh}]\, /2 \qquad Eq\ 59$$

with τ_0 indicating the shear stress due to clear water and τ_h the additional shear stress due to suspended particles. Note that:

$$\tau_0 + \tau_h = \frac{(i_0 + i_h)\, \rho_w\, g\, A}{S}, \qquad \text{with } (i_0 + i_h) \text{ as shown in } Eq\ 61.$$

Wilson made the calculations for determining C_{dc}/C_d for values of V between 1 and 8 m/s. The results are given in Wilson's Figure (6), and the mean line (C_c/C in Wilson's notation) is accepted, after some checking by the authors.

Calculations reported here are for the velocity $V = 5.0$ m/s. The parameters that are constant throughout the calculations were determined as follows:

From measurements made on Wilson's Figure (6), C_{dc}/C_d was calculated as 0.2338 at $V = 5.0$ m/s. Then*:

$$C_{dc} = (C_{dc}/C_d) \times C_d = 0.2338 \times 0.18 = 0.04208$$

$$C_{dh} = C_d - C_{dc} = 0.18 - 0.04208 = 0.1379$$

$$C_{dc}/C_b = 0.04208/0.6 = 0.07013$$

$$1 - (C_{dc}/C_d) = 0.7662$$

$$1 - (C_{dc}/C_b) = 0.9299$$

$$C_1 = 0.1483 \qquad \text{(see } Eq\ 43\text{)}$$

$$\rho_1/\rho_w = 1.2447. \qquad \text{(see } Eq\ 19\text{)}$$

* Not all the digits given are significant.

Calculations of the Force Balance Equations, for $V = 5.0$ m/s

At the outset of the calculations the least possible value of β can be determined from Equation 41, because V_2 must be less than, or equal to, V, by assuming that $V_2 = V$, i.e., by assuming that $\frac{C_{dc}}{(1-a)\,C_b} = 1$.

It follows that

$$\frac{\beta - \sin\beta\cos\beta}{\pi} = (1-a) = \frac{C_{dc}}{C_b} = 0.07013.$$

By successive trials, or interpolation from a graph, it can be determined that $\beta = 0.72$ is slightly greater than the least possible value.

For $\beta = 0.72$, widely different values of i_1 and i_2 were found. By successive trials, i_1 and i_2 were found to be approximately equal for $\beta = 1.23$. The values of i_1 and i_2 that were determined, are shown in Table 3.

TABLE 3

Values of i_1 and i_2 at $V = 5.0$ m/s, for a range of β values

β (deg)	β (rad)	i_1	i_2	$i = [ai_1 + (1-a)i_2]$
41.3	0.72	0.043	0.576	0.081
45.8	0.80	0.050	0.449	0.088
57.3	1.00	0.092	0.271	0.123
67.0	1.17	0.155	0.211	0.169
68.8	1.20	0.170	0.203	0.179
70.5	1.23	0.186	0.196	0.189

The results of a set of calculations, made with $\beta = 1.230$, and $V = 5.0\ m/s$, are summarized in Table 4. Hence $i = ai_1 + (1-a)i_2 = 0.189\ (\frac{m}{m})$ with values of i_1 and i_2 obtained from Equations 44 and 45, respectively.

To sufficient accuracy the computed result, on the basis of Shook's Model B, is

$$i_B = 0.190\ (\frac{m}{m}).$$

TABLE 4

Results of sample force balance calculation, with $\beta = 1.230$ and $V = 5.0$ m/s

		see:			see:
$1 - a =$	0.29125	Eq 51	$V_2 =$	1.2039 (m/s)	Eq 41
$a =$	0.70875		$V_1 =$	6.5599 (m/s)	Eq 42
$\phi_1(\beta) =$	0.16914	Eq 54	$V_1 - V_2 =$	5.3560 (m/s)	
$\frac{S_1}{A_1} =$	8.5852 $(\frac{1}{m})$	Eq 46	$\frac{\tau_1}{\rho_w} =$	87.041×10^{-3} $(\frac{m}{s})^2$	Eq 56
$\frac{S_2}{A_2} =$	13.4428 $(\frac{1}{m})$	Eq 47	$\frac{\tau_{12}}{\rho_w} =$	$255.303 \times 10^{-3}(\frac{m}{s})^2$	Eq 57
$\frac{S_{12}}{A_1} =$	4.2328 $(\frac{1}{m})$	Eq 48	$\frac{\tau_2 f}{\rho_w} =$	$2.932 \times 10^{-3}(\frac{m}{s})^2$	Eq 58
$\frac{S_{12}}{A_2} =$	10.3006 $(\frac{1}{m})$	Eq 49	$\frac{\tau_{2\eta}}{\rho_w} =$	$333.650 \times 10^{-3}(\frac{m}{s})^2$	Eq 52
$\frac{S_2}{A} =$	3.9152 $(\frac{1}{m})$	Eq 50	$\frac{\overline{\tau}_2}{\rho_w} =$	$336.582 \times 10^{-3}(\frac{m}{s})^2$	Eq 28

Due to the uncertainties inherent in both the theory, and the data, no further calculations are needed.

* * * * * *

The basis for Wilson's Model A is to write

$$i = i_0 + i_h + i_\eta \qquad Eq\ 60$$

where:

- i_0 is the headloss for clear water, flowing in the whole pipe at velocity V,
- i_h is the increment of head loss due to the concentration C_{dh} of suspended particles (as delivered), when the slurry is flowing in the whole pipe, and
- i_η is the increment of head loss due to sliding friction between the contact-load (in situ, in the bed), and the pipe.

The head losses i_0 and i_h correspond to the shear stresses τ_0 and τ_h, respectively, in Equation 59; also noting that both τ_0 and τ_h act along S, we obtain:

$$A(i_0 + i_h)\rho_w g = S(\tau_0 + \tau_h)$$

from which it follows that

$$i_0 + i_h = \frac{2}{gD} C_{f0} V^2 [1 + (s-1) C_{dh}]$$

$$= \frac{2}{9.81 \times 0.4} \times 3.25 \times 10^{-3} \times 5.0^2 [1 + (2.65 - 1)0.1379] = 0.051 \qquad Eq\ 61$$

Note that $j = i_0 + i_h = i_0 + i_0(s-1)(C - C_c)$ per page A1-4 of Wilson (1976), with $i_0 = \frac{f_0 V^2}{2gD} = \frac{2C_{f0}V^2}{gD}$.

Similarly the head loss i_η corresponds to the shear stress $\tau_{2\eta}$, calculated from Equation 52. Hence, from basic considerations,

$$A i_\eta \rho_w g = S_2 \tau_{2\eta} \qquad Eq\ 62$$

since i_η is a normal stress acting over the total area A of the flow, and $\tau_{2\eta}$ is a shear stress acting only over the area S_2 of the pipe wall.

Thus,

$$i_\eta = \frac{S_2}{gA} \frac{\tau_{2\eta}}{\rho_w} = \frac{3.9152}{9.81} \times 333.650 \times 10^{-3} = 0.133. \qquad Eq\ 63$$

Then, by Equation 60, using $\beta = 1.230$,

$$i_A = 0.051 + 0.133 = 0.184 \ \left(\frac{m}{m}\right)$$

on the basis of Wilson's Model A.

The component i_η is actually calculated from exactly the same factors for both models. The difference between i_A and i_B is due only to terms dealing with hydraulic flow. In Wilson's Model A they are i_0 and i_h, and their sum may be designated as $(i_0 + i_h)_A$, whose value was calculated through Equation 61.

The counterpart of this sum, according to Shook's Model B, may be designated as $(i_0 + i_h)_B$, although the individual terms that make it up are not directly related to i_0 and i_h of Wilson's Model A.

Because a solution of Shook's Model B is independent of τ_{12}, as shown later in this section, and because $(i_0 + i_h)_B$ depends only on τ_1 and τ_2 (which is only part of $\overline{\tau}_2$), one may write

$$g(i_0 + i_h)_B = a \frac{\tau_1}{\rho_w} \frac{S_1}{A_1} + (1-a) \frac{\tau_{2f}}{\rho_w} \frac{S_2}{A_2} \qquad Eq\ 64$$

Using the values given in Table 4, we find $(i_0 + i_h)_B = 0.055$, which is to be compared with $(i_0 + i_h)_A = 0.051$. The difference between $(i_0 + i_h)_A$ and $(i_0 + i_h)_B$ accounts for the

difference betwen i_A and i_B. The differences referred to here are not exactly equal, but this lack of equality results solely from inaccuracies in the calculations.

By using Figure (7) of Wilson (May 1976), the value of i for $V = 5\ m/s$ is found to be $i'(7) = 0.108$.

The value found by the authors, by following the procedure used by Wilson from his Figure (3) (May 1976, p. A1-10, para. 4), is $i(3) = 0.113$, a value close to $i'(7)$, as expected. We adopt the mean as $i(7) = 0.110$

The reason for the discrepancy between $i(7)$, and Wilson's $i_A(= 0.184)$, or Shook's $i_B(= 0.190)$, is not known, as indicated earlier in this section. Several possible sources of error, either on the part of the present authors, or of Wilson, were explored, but to no avail.

In the course of this work a study was made of the exact meaning of Wilson's parameter ζ, and a divergence between Eyler and Wilson was recognized. Some of the results of the study may be found to be instructive. At the least, a few sources of confusion in the literature may be neutralized. In addition, a simple method was found for determining ζ, an important parameter of both Model A or Model B, when fitting Shook's Model (B) to data.

Parameter C_1'

Eyler and Lombardo (1980, p. 3.36), and Eyler et al. (1982, p. 5.42) introduce the parameter C_1, which will be referred to here as C_1', to avoid confusion with our concentration parameter C_1. Equation (5.44) of Eyler et al. (1982), can be written as

$$\tau_{12} = \tau_i = C_1' C_{fi} (V_1 - V_2)^2 \rho_1 / 2 \qquad Eq\ 65$$

because, as noted earlier, $f_i = 4C_{fi}$ and V_a becomes $(V_1 - V_2)$ in the case of a moving bed (see Eyler et al. 1982, section 5.7.3). For ρ_1, note our Equation 19.

Eyler et al. (1982) regard C_{fi} as the friction factor of the bed surface, with all particles fixed relative to each other. They intend the factor C_1' to take care of the effect of particles *saltating*† on the surface.

Wilson, on the other hand, has clearly intended C_{f12} to be the total effective friction factor of the bed surface. Thus, $C_{fi} \neq C_{f12}$. For Wilson

$$C_{f12} = \zeta C_{f0} \qquad Eq\ 66$$

† *Saltation* refers to particles that are dislodged from the bed surface by forces exerted by the flow above it.

for Eyler et al. (1982),

$$C_{f12} = C_1' C_{fi} = C_1' \xi C_{f0} \qquad Eq\ 67$$

where Eyler's definition of ξ is

$$C_{fi} = \xi C_{f0}. \qquad Eq\ 68$$

Thus,

$$\zeta = C_1' \xi. \qquad Eq\ 69$$

That Eyler et al. (1982, p. A.15) use Equation 68 is confirmed by their Equation (A.17), which reads

$$\phi_2 = C_1' \xi (1 - V_R)^2 \sin\beta \qquad Eq\ 70$$

Since the 1980 and 1982 reports of Eyler and his colleagues are not completely consistent, the following points should be noted:

The 1980 Equation (4.2) reads

$$f_i = 2f = 2\ [2log_{10}(D/d) + 1.14]^{-2} \qquad Eq\ 71$$

where f is the factor given by the Nikuradse formula. Thus, Wilson's factor 2 has been adopted, but C_1' is also used as an additional multiplier.

The 1982 Equation (5.38) reads

$$1/\sqrt{f_i} = -0.86\, ln_e\, (\epsilon/D) + 1.14$$

which is equivalent to‡

$$f_i = f = [2\ log_{10}(D/\epsilon) + 1.14]^{-2} \qquad Eq\ 72$$

The factor 2 has been dropped, but also (D/d) has been replaced by (D/ϵ), where*

$$(\epsilon/D) = 1.72\,(d/D) \qquad Eq\ 73$$

Equation 72 is equivalent to

$$f_i = v\ [2log_{10}\ (D/d) + 1.14]^{-2} \qquad Eq\ 74$$

where $v = v(d/D)$ is a variable function of (d/D) that replaces Wilson's factor 2.

Eyler and Lombardo (1980, p. 3.36) say that they could reproduce the results of Wilson (May 1976) by using $C_1' = 1.25$, but that Judge (1979) reported that, in their

‡ Since $0.43\, ln\ a = log_{10}\ a$, and $-ln(\frac{\epsilon}{D}) = ln(\frac{\epsilon}{D})^{-1}$.
* Eyler's 1982 Equation (5.41).

words, "the original model developed by Wilson effectively used $C_1' = 1$." Since Eyler and Lombardo included an equation equivalent to our Equation 71 in the same paper, the total multiplier of f would be $2C_1' = 2.50$. The remark attributed to Judge can only mean:

— either, that the "original model" dealt only with a fixed bed with no saltation; Wilson (1971, p. 1667, following Equation 6) notes that, even with a "fixed" bed, an increased friction factor occurs if saltating particles move over the surface; or

— that Judge assumed that Eyler's ξ equals Wilson's ζ, for a mobile bed surface, which leads to $C_1' = 1$, by Equation 69.

However, it can be seen that Eyler et al. (1982, Section 5.8.2, p. 5.51) only treated flow with suspended particles in a tentative way, unsuccessful in its results. It is highly unlikely that Eyler and Lombardo (1980, p. 3.36) include Wilson's illustrative example, treated here, in the remark concerning $C_1' = 1.25$.

It may be mentioned, by the way, that Shook (1981) gives

$$C_{f12} = 2/[4\ log_{10}(D_{el}/d_m) + 3.36]^2 \qquad [A11]$$

which includes Wilson's factor 2 in the numerator. But our Equation 71, with f_1 replaced by f_{12}, is equivalent to

$$C_{f12} = 2/[4\ log_{10}(D_{el}/d_m) + 2.28]^2 \qquad Eq\ 75$$

since $C_{f12} = f_{12}/4$. This is the form given here in Equation 55. The difference between it and Shook's [A11] appears to be the result of an error.

* * * * * *

Returning now to our calculations, it is possible, by including a parameter C_1', to find a solution for Shook's Model B that will make i_B agree with $i(7)$ of Wilson's Model A. As a motivation for doing this, let us suppose that $i(7)$ represents an experimental result, with which we want the result from Shook's Model B to agree. We are prepared to change the factor 2 used by Wilson in calculating C_{f12} to a factor $2C_1'$ (in the sense in which Eyler and Lombardo introduced C_1' in 1980).

From Equations 63 and 52, we find that

$$i_\eta = \frac{S_2}{gA}\frac{\tau_{2\eta}}{\rho_w} = j_p \times \phi_1(\beta). \qquad Eq\ 76$$

TABLE 5

Results of a trial solution for Model B, with $\beta = 0.925$ and $V = 5.0$ m/s

		see:			see:
$1 - a =$	0.14144	Eq 51	$V_2 =$	2.4791 (m/s)	Eq 41
a =	0.85856		$V_1 =$	5.4153 (m/s)	Eq 42
$\phi_1(\beta) =$	0.07701	Eq 54	$V_1 - V_2 =$	2.9362 (m/s)	
$\frac{S_1}{A_1} =$	8.2180 $(\frac{1}{m})$	Eq 46	$\frac{\tau_1}{\rho_w} =$	$59.317 \times 10^{-3}\ (\frac{m}{s})^2$	Eq 56
$\frac{S_2}{A_2} =$	20.8163 $(\frac{1}{m})$	Eq 47	$\frac{\tau_{12}}{\rho_w} =$	$76.729 \times 10^{-3}(\frac{m}{s})^2$	Eq 57
$\frac{S_{12}}{A_1} =$	2.9609 $(\frac{1}{m})$	Eq 48	$\frac{\tau_{2f}}{\rho_w} =$	$12.431 \times 10^{-3}(\frac{m}{s})^2$	Eq 58
$\frac{S_{12}}{A_2} =$	17.9722 $(\frac{1}{m})$	Eq 49	$\frac{\tau_{2\eta}}{\rho_w} =$	$203.210 \times 10^{-3}(\frac{m}{s})^2$	Eq 52
$\frac{S_2}{A} =$	2.9444 $(\frac{1}{m})$	Eq 50	$\frac{\overline{\tau}_2}{\rho_w} =$	$215.641 \times 10^{-3}(\frac{m}{s})^2$	Eq 28

Thus, $\phi_1(\beta) = \frac{i_\eta}{j_p}$.

For the purpose of this exercise we let‡

$$i_\eta = i'_\eta = i(7) - (i_0 + i_h)_A = 0.110 - 0.051 = 0.059.$$

Then

$$\phi(B') = \frac{i_\eta}{j_p} = \frac{0.059}{0.792} = 0.0745$$

or, from Equation 54,

$$\frac{\sin\beta' - \beta'\cos\beta'}{\pi} = 0.0745.$$

By a linear interpolation in the series of values of $\phi_1(\beta)$ already calculated, we find that $\beta' = 0.925$ ($\approx 53°$), which is accepted as sufficiently accurate for the purpose, since

$$\phi_1(0.925) = 0.770.$$

This means that the hypothetical experimental value $i'_\eta = 0.059$ is compatible only with $\beta' = 0.925$ ($\approx 53°$), according to Equation 76 (note that Equation 76 applies to both Model A and Model B). A trial solution was calculated for Model B, using $\beta' = 0.925$; the quantities that differ from those given in the previous example are shown in Table 5.

‡ For $(i_0 + i_h)_A$ see Equation 61; for $i(7)$ see Equation 64.

Hence

$$10^3 i_1 g = 10^3 \left[\frac{\tau_1}{\rho_w} \frac{S_1}{A_1} + \frac{\tau_{12}}{\rho_w} \frac{S_{12}}{A_1} \right] = 487.467 + 227.187 = 714.654$$

thus,

$$i_1 = 0.0728.$$

Also

$$10^3 i_2 g = 10^3 \left[\frac{\overline{\tau}_2}{\rho_w} \frac{S_2}{A_2} - \frac{\tau_{12}}{\rho_w} \frac{S_{12}}{A_2} \right] = 4488.848 - 1378.989 = 3109.859$$

thus,

$$i_2 = 0.3170$$

and,

$$i = a i_1 + (1 - a) i_2 = 0.1073.$$

This value of i is expected to be close to $i(7) = 0.110$, because of the way that β' was chosen. However, the trial solution is not correct, since i_1 and i_2 are not equal.

The above solution effectively uses $C_1' = 1$ in Equation 65. From this equation we see that if a different value, x, is used for C_1', then τ_{12} will become $(x\,\tau_{12})$. Thus the above numerical equations for calculating $i_1$and i_2 will become

$$10^3 i_1 g = 487.467 + 227.187\ x$$

and,

$$10^3 i_2 g = 4488.848 - 1378.989\ x$$

By equating i_1 and i_2 we obtain an equation for x, i.e.,

$$4488.848 - 487.467 = (1378.998 + 227.187)\ x$$

which gives

$$x = 2.49125. \qquad Eq\ 77$$

When we use this value of x in the equation for i_1 and i_2, we find that

$$i = i_1 = i_2 = 0.1074.$$

This is the same value as the original one, found with $C_1' = 1$. In fact it may be noted that i is independent of τ_{12}; this can be shown as follows:

$$\begin{aligned} ig = a i_1 g + (1 - a) i_2 g &= a \left[\frac{\tau_1}{\rho_w} \frac{S_1}{A_1} + \frac{\tau_{12}}{\rho_w} \frac{S_{12}}{A_1} \right] + (1 - a) \left[\frac{\overline{\tau}_2}{\rho_w} \frac{S_2}{A_2} - \frac{\tau_{12}}{\rho_w} \frac{S_{12}}{A_2} \right] \\ &= a \frac{\tau_1}{\rho_w} \frac{S_1}{A_1} + (1 - a) \frac{\overline{\tau}_2}{\rho_w} \frac{S_2}{A_2} + \frac{\tau_{12} S_{12}}{\rho_w} \left[\frac{a}{A_1} - \frac{(1 - a)}{A_2} \right]. \end{aligned}$$

But,

$$\frac{A_1}{A} = a \quad \text{and} \quad \frac{A_2}{A} = \frac{A - A_1}{A} = (1 - a)$$

hence

$$\frac{a}{A_1} = \frac{1}{A} \quad \text{and} \quad \frac{(1-a)}{A_2} = \frac{1}{A}$$

consequently,

$$\left[\frac{a}{A_1} - \frac{(1-a)}{A_2}\right] = \left[\frac{1}{A} - \frac{1}{A}\right] = 0$$

i.e.,

$$i \text{ is independent of } \tau_{12}.$$

Shook (1981, p. 10) notes, with reference to his Figure (13), that "the interfacial friction factor C_{f12} [to which, per Equation 57, τ_{12} is proportional], is of secondary importance," as far as the value of i is concerned. For a given β this is true, but an i-value is only acceptable to the model, if the balance condition ($i_1 = i_2$) is fulfilled, and this is determined by τ_{12}. If the condition is not fulfilled, and τ_{12} is fixed, then β must change, and thus i. In that sense an acceptable value of i is strongly dependent on τ_{12}, or on C_{f12}.

But as shown above, a fixed value of β can be derived from experimental results (see Equations 76, et seq.). If other parameters (excepting τ_{12}) are also fixed, the model can only fit the data if a new value of τ_{12} is adopted.

Since† $C_{f12} = \zeta C_{f0}$, and since our original ζ-value is 4.4, the end result of the foregoing exercise is that, to fit Shook's Model B to Wilson's $i(7) = 0.110$, it is necessary to use‡

$$\zeta' = \zeta x = 4.4 \times 2.5 = 11$$

Also: $C_{f12} = \zeta' C_{f0}$, which is the value required for calculating a new value for τ_{12}¶.

The procedure for finding x is very simple, once a calculation has been done, using any arbitrary starting value of ζ. It could be useful for calculating ζ-values appropriate to a series of sets of data, assuming that all the other parameters of the model are well known (or arbitrarily chosen). Thus, an empirical function of V could be found for ζ, i.e., $\zeta = \zeta(V)$.

Because, in fact, several parameters are not well established theoretically, in a model that does not closely mirror the physical reality of the flow, this relation simply eliminates one degree of freedom in the adjustment of parameters. It might help the empirical development of the model.

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† See Equation 66.
‡ With $x = 2.5$, per Equation 77.
¶ See Equations 27 and 57.

DISCUSSION OF A PLAN FOR DATA ACCUMULATION ON THE BASIS OF DIMENSIONAL ANALYSIS

Shook (1976, p. 21), Shook et al. (1981, p. 91) and Eyler et al. (1982, p. 2.9) express the need for experimental data on large-particle slurry flows in large pipes, which are essential for the testing and development of design equations and, in particular, of the Wilson model. As other models may also merit testing, and as the experimental work is expensive, a generalized approach to any opportunity for such work should be discussed and developed in anticipation, by members of the joint Canada/FRG 'Coarse-Slurry Working Group'.

Because of the complexity peculiar to hydraulic flow, indicated very briefly in Appendix A, any mathematical flow model is likely to be semi-empirical for the foreseeable future. For this reason practical work on a particulate solid (whose slurry flow properties are not already well known) is likely to continue to be essential to the planning of any pipeline installation.

Nevertheless, it is proposed that a fruitful approach to the preliminary work may consist in an attempt to provide physical models that fulfil, as accurately as possible, similarity to the proposed prototype flow.

It is common, when investigating the slurry flow properties of a given particulate solid, to compare runs of the same material in pipes of different sizes. If d is the characteristic size of the particles, and D is the pipe diameter, a change in D involves a change also in D/d. Since D/d is a characteristic variable of the flow, a condition of similarity between the flows is violated. It should be noted that the significance of D/d must increase as the ratio decreases, i.e., when larger particles are of interest.

The mathematical model under test represents an attempt to deal with both the scale change in D, and the change in D/d, but when some effects of the changes are not fully understood, interpretation of results may be difficult. The effects are *confounded.*

By scaling d (by crushing or sieving, or both) to provide the same D/d ratio in the pipe flows compared, the corresponding criterion of similarity is fulfilled. This gives more specific information on the effect of the change in D (and d), unaffected by a change in D/d.

As shown in Appendix B, the variable Ξ (or equivalently X, a Reynolds number, depending on the group of basic parameters chosen), cannot be kept constant if the criterion associated with the variable Y is fulfilled. The same applies to $X_6 = k/d$ (proportional to pipe roughness ratio), unless extraordinary measures are taken.

However, the effects of Ξ and X_6 may not be of major importance under flow conditions with large particles in large pipes. A direct measurement of the combined effect of these variables, when D changes, would appear to provide useful experimental information. If only a minor effect exists, the results can allow an extrapolation to the prototype scale, approximately, as follows.

Large-diameter pipelines are now available for laboratory work. Two different pipe sizes, as close as possible to the prototype size, should be chosen for the model work.

Let the diameter of the prototype pipe be D', and let the diameters of the two model pipes be D_1'', and D_2'', where $D' > D_1'' > D_2''$. Let $C' = C_1'' = C_2''$, and $D'/d' = D_1''/d_1'' = D_2''/d_2''$, i.e., $Z' = Z_1'' = Z_2''$, since $Z = D/d$.

Suppose that the data are plotted as in Figure (B-5) (see Appendix B), where D' and D'' in that figure now become D_1'', and D_2''. If Y is plotted against D for some chosen fixed value of $\overline{\overline{\Pi}}_V$, the plotted points a and b are shown as in Figure 10. The line $a - b$ is extrapolated to c, and $b - d$ is drawn parallel to the D axis.

The point e represents the coordinates (D', Y') for the prototype, with

$$(\overline{\overline{\Pi}}_V)' = (\overline{\overline{\Pi}}_V)_1'' = (\overline{\overline{\Pi}}_V)_2''.$$

If there were no significant effect of Ξ and of X_6 (a case represented by Figure (B-4), rather than by Figure (B-5) in Appendix B), line $a - b - c$ would be parallel to the D axis, lines $b - d$ and $b - c$ would coincide‡, and point e would lie on line $a - b - c$. This represents conditions of a *fully turbulent* flow, in which case $Y' = Y_1'' = Y_2''$ (see Appendix B).

If a significant effect of Ξ and of X_6 exists, as represented in Figure 10, then the relative effect should decrease¶, as D increases. Therefore, the point e is placed as shown, somewhere in the interval of the ordinate through D', defined by the lines $b - d$ and $b - c$.

The uncertainty range of point e represents the uncertainty of

$$Y' = \frac{i'}{4(s-1)}\left(\frac{D'}{d'}\right) = K\, i'$$

where, K is a calculable constant. This range of uncertainty does not seem to have been determined, in any previous work, by the direct experimental process described, for comparison with the uncertainty of extrapolation by a mathematical model.

Aside from the direct practical benefit of extrapolation, as described in Figure 10 (a hypothetical benefit, subject to investigation), a second important consideration also

‡ Because Y would be the same for D', D_1', and D_2''.
¶ Because the flow becomes more turbulent, as D increases.

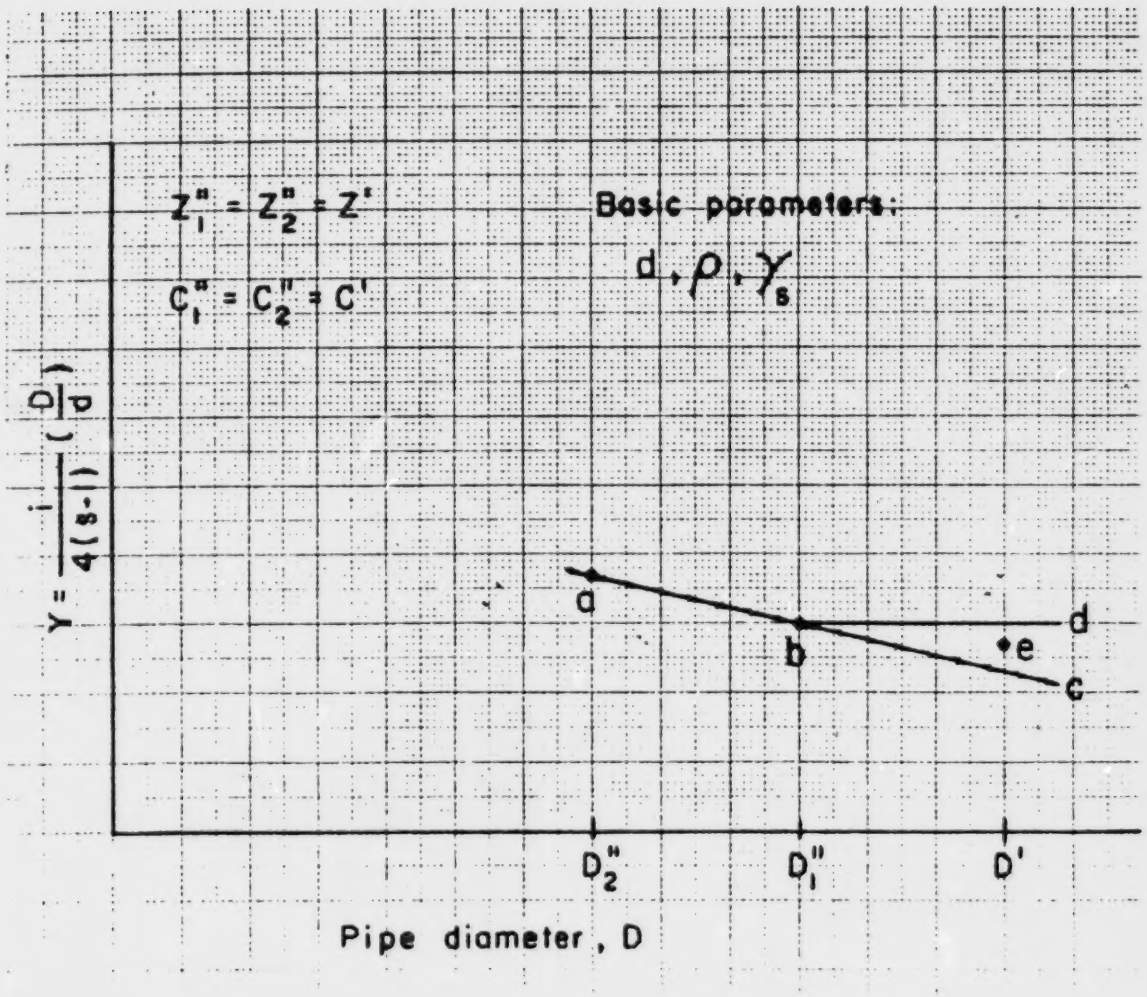

Fig. 10 — Plot of Y against D for a chosen value of $\overline{\overline{\Pi}}_V = \frac{V}{\sqrt{D\,g\,(s-1)}}\sqrt{\frac{D}{d}}$

provides a strong motive for this proposal (i.e., to use similarity criteria). This consideration involves the fact, that it is always hard to compare results from different laboratories, and quite especially so when the particulate solids involved are not identical. A model can be subjected to a much more searching test of accuracy when it is applied to two (or more) sets of data, between which only a single, controllable variable differs. Therefore, the reliability of a model should first be established in the simplest situations, before a wider versatility is sought.

Adoption of the approach described would encourage generation of well-defined, closely related sets of data, that would be most valuable for the testing of any model. Moreover, an improvement of the comparability of data from different laboratories could

thereby also result. This would be a bonus of particular importance in cases like the present one, where cooperation not only of different laboratories, but also of different countries is involved.

Notes

In connection with the foregoing study, a crude test was made of the relative importance of the variables X and Y as criteria of similarity between flows in pipes of large diameter. Under the conditions of the experiments proposed above, the criteria reduce to

$$V' = V'' \, D'' / D' \qquad \text{criterion } (X)$$

and,

$$V' = V'' \, \sqrt{D' / D''} \qquad \text{criterion } (Y)$$

Some data listed by Eyler et al. (1982, Appendix B) were used for the test.

Data due to Worster and Denny are shown in Table 6. Here V_C is *critical velocity*, corresponding to the minimum i of the $i - V$ curve.

TABLE 6

Data due to Worster and Denny, as listed by Eyler et al.

d (mm)	D (mm)	C_d %	d/D	V_C (m/s)
38.0	150	5.0	0.25	1.067
38.0	150	10.0	0.25	1.31
38.0	150	15.0	0.25	1.46
12.5	76	5.0	0.167	0.76*
12.5	76	10.0	0.167	0.85*
12.5	76	15.0	0.167	0.92*
				* = V_C''

Source: Eyler et al., 1982, Appendix B, p. B.3.

These data consist of two sets, using similar coarse solids for transport in two large pipes. Unfortunately, the values of d/D, belonging to the two sets are not as nearly equal as one would wish, but it is interesting to calculate V'_C values for the larger pipe on the basis of V''_C measured in the smaller pipe.

V_C is the velocity at the minimum i of an $i - V$ curve, and we assume that V_C values occur at similar flows, when conditions for dynamic similarity are fulfilled. Calculations were done, using the above equations in the following forms

$$V'(X) = V''_C \, D'' / D' \qquad \text{criterion } (X)$$

and,

$$V'(Y) = V''_C \sqrt{D' / D''} \qquad \text{criterion } (Y)$$

To compare $V'(X)$ and $V'(Y)$ values with the V_C values observed in the larger pipe, we present the results in Table 7.

TABLE 7

Comparison of V' values with V_C values

$V'(X)$	$V'(Y)$	V_C	$V_C - V'(Y)$
0.38	1.07	1.067	0 %
0.43	1.19	1.31	10 %
0.47	1.29	1.46	13 %

It is seen that the prediction on the basis of criterion (Y) is good, but not on the basis of criterion (X).

One further example can be drawn from data involving the largest pipes listed, due to Gödde (last item) and Weber and Gödde (first item). These data are shown in Table 8.

It is noted that d / D values are almost equal. Results of calculations, similar to those undertaken earlier on the basis of data shown in Table 6, are shown in Table 9. Again, the prediction for criterion (Y) is better than that from criterion (X).

The evidence comes from random selection, and is not very satisfactory, but the results are interesting all the same. Scaling in accordance with the Y criterion (Froude) seems to be good enough to encourage the use of the hypotheses stated in Appendix B (ahead of Eq (B-16) and following Eq (B-26)), and used in this section.

TABLE 8

Data due to Gödde and Weber + Gödde, as listed by Eyler et al.

d (mm)	D (mm)	C_d %	d/D	V_C (m/s)
8.68	150	15.0	0.058	1.79*
12.50	200	15.0	0.063	2.73
				*=V″

Source: Eyler et al., 1982, Appendix B, p. B.2.

TABLE 9

Comparison of V' values with V_C values

$V'(X)$	$V'(Y)$	V_C	$V_C - V'(Y)$
1.34	2.07	2.73	24 %

STUDIES ON VARIOUS TOPICS RELATING TO THE WILSON MODEL

Suggestion re. " Problem of the Applicability of the Vertical Transmitted Stress Component $\tau_i / \tan\phi$ "

Shook (1983, pp. 7 and 20) comments on the term $\tau_i / \tan\phi$, equivalent to $\tau_{12} / \tan\phi$ in equation (A13) of Shook (1981), for the vertical transmitted stress component. This term, $\tau / \tan\phi$, was introduced by Wilson (1970, p. 3) with reference to Bagnold (1957).

It is suggested here that the term in question resulted from a misinterpretation of Bagnold's work, and that it should not be included in the calculation of the pressure exerted by the bed on the pipewall. A brief justification is offered hereunder.

In an experimental study of flowing mixtures of fluid and grains, Bagnold (1954) developed a concept of the grains regarded as a separate fluid. The experimental method involves particles equal in density to the fluid and the stress P (Figure 11) is measured in a horizontal direction. However, the concept is also applicable to particles whose density is greater than that of the fluid, and the stress P can arise in the vertical direction (Bagnold, 1955).

Suppose that the mixture undergoes simple shear, as illustrated in Figure 11.

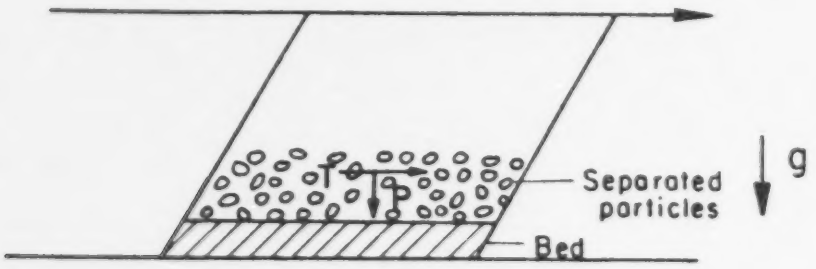

Fig. 11 — Schematic illustration of slurry flow in pipe

It is assumed that some of the particles of the mixture exist above the bed, because they are dislodged by a shear force exerted by the fluid. The particles are in saltation.

In addition, collisions of particles returning to the bed may dislodge other particles. On the average, a population of separated particles is maintained above the bed. Collisions between separated particles also take place.

When layers of particles (on the average) shear past each other, a shear stress T is conceived to be exerted between layers, due only to encounters between grains. Because of these encounters a vertical normal stress P is also exerted between the particles. Bagnold considered

$$T/P = \tan\alpha$$

as the dynamic analogue of a static friction coefficient, but not identical to the internal friction of the bed.

The stress P illustrated in Figure 11 represents a vertical, expansive stress acting between particles only. The stress does not add a new load to the bed, but transmits to the bed the net gravitational force acting on the particles, i.e., the total submerged load of the separated particles is applied to the bed just as if the separated particles formed part of the bed.

Correct references have been made to Bagnold's concept, which explains that the full weight of separated particles can be carried by the bed.

But, other references appear to assume that when a layer of particles (even of minimal thickness) is mobilized at the surface of the bed by a shear stress τ applied by a fluid, there exists an associated normal stress P applied to the bed. The relation $P = \tau/\tan\phi$, where ϕ is the angle of repose associated with the bed, cannot be justified. It is not the same relation $P = T/\tan\alpha$, as indicated above.

A shear stress applied to the surface of a bed cannot exert a normal stress, other than the gravitational stress of separated particles.

Notes

(1) If an expansive stress P is acting in a pipe, and if the upper separated particles should collide with the upper wall, the reaction of the wall could cause an increased pressure on the bed. This type of reaction was the basis for Bagnold's (1954) experiment.

(2) Particles are purely in saltation only when they are not partly supported by a dynamic process of the fluid. If the

fluid does support part of the submerged weight of the particles by turbulence, or by other effects, then the load of the bed is diminished to that extent.

(3) The clearest statement of Bagnold's concept of the bed load due to dispersed or to separated particles, is given by Bagnold (1957, p. 250).

(4) Wilson (1975, p. 5) makes the following statement: *Since the intergranular pressure p_s will be zero at this surface, it can be used as the datum for z in Equation 2.* This seems to be tantamount to withdrawal of the $\tau/\tan\phi$ term from the Wilson model. However, reference continues to be made to Wilson (1970) without comment, although the term does not appear again.

Buoyancy Effect on Contact Particles

Calculation of the pressure difference $P_B - P_A$

On examining Figures (6) and (7) of Shook (1981), and Figures (11) and (12) of Shook et al. (1982a), we find the situation illustrated in Figure 12(a).

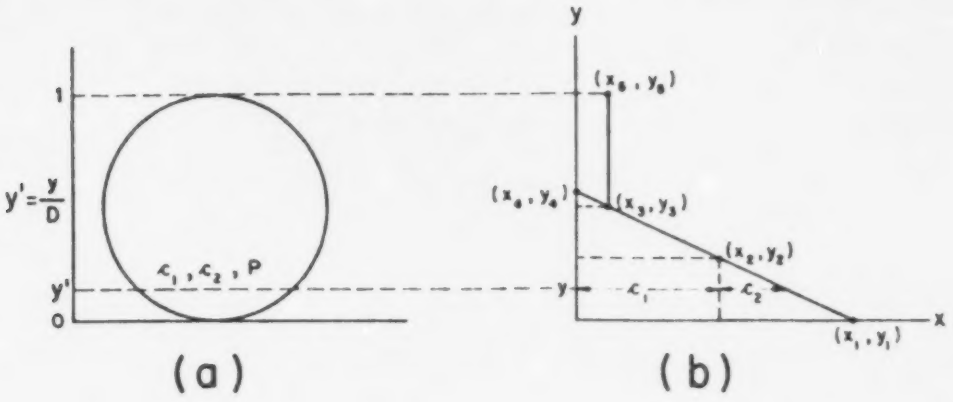

Fig. 12 — Schematic illustration of calculation techniques:
(a) where y' is defined by Shook et al.
(b) where primes are dropped for convenience

At height y, the particles in transport have the following concentrations:

c_1 = local volume fraction of particles supported by water

c_2 = local volume fraction of particles supported by the pipe wall

hence,

$c_f = 1 - c_1 - c_2$ =local volume fraction of water.

The water, plus the particles supported by the water, is equivalent to a slurry occupying the volume fraction

$$c_f + c_1 = 1 - c_2$$

hence the density** of the slurry is

$$\rho_{sl} = \frac{\rho_f c_f + \rho_s c_1}{1 - c_2} = \frac{\rho_f(1 - c_1 - c_2) + \rho_s c_1}{1 - c_2}$$

$$= \frac{\rho_f(1 - c_2) + (\rho_s - \rho_f)c_1}{1 - c_2} = \rho_f \frac{1 - c_2 + (s-1)c_1}{1 - c_2}$$

$$= \rho_f \left[1 + \frac{(s-1)c_1}{1 - c_2}\right].$$

The pressure gradient at level y is given by

$$\frac{\partial P}{\partial y} = -g\,\rho_f \left[1 + \frac{(s-1)c_1}{1 - c_2}\right] \qquad \text{(due to slurry)}$$

With reference to the top of the pipe:

$$P(y) = -g\,\rho_f \int_D^y dy - g(s-1)\rho_f \int_D^y \frac{c_1}{1 - c_2} dy$$

$$= P_1(y) + P_2(y)$$

Thus the pressure difference component due to $P_2(y)$ between the top and the bottom of the pipe is given by

$$P_2 = P_2(0) = +^{(*)} g(s-1)\rho_f \int_0^D \frac{c_1}{1 - c_2} dy.$$

Let $y = y'D$, as used on Figure 12(a). Hence, $dy = Ddy'$, and

$$P_2 = g(s-1)\rho_f D \int_0^1 \frac{c_1}{1 - c_2} dy'$$

** ρ_s indicates the density of the solid particles. If coal (or some other porous material) is involved, ρ_s indicates the density of water-saturated particles. Also, $s = \rho_s/\rho_f$, and c_f, c_1, and c_2 are concentrations per unit volume of mixture.

(*) + sign, because the $\int$ is from 0 to D.

thus,

$$\frac{P_2}{g(s-1)\rho_f D} = I = \int_0^1 \frac{c_1}{1-c_2} dy' = I_1 + I_2 + I_3, \qquad \text{as defined below.}$$

On Figure 12(b), x coordinates represent particle volume fractions $(c_1 + c_2)$, and y coordinates (dropping the primes) represent scaled heights $\frac{y}{D}$. Coordinates of critical points are marked:

I_1 is integrated from $y = 0$ to y_2

I_2 is integrated from $y = y_2$ to y_3

I_3 is integrated from $y = y_3$ to 1.

Let the slope of the lower part of the concentration distribution be m (absolute value), i.e.,

$$m = \left| \frac{y_4 - 0}{0 - x_1} \right| = \frac{y_4}{x_1}$$

To calculate I with reference to Figure (11) of Shook et al (1982a), we use:

$$x_1 = 0.700 \qquad y_1 = 0.000$$

$$x_2 = 0.120 \qquad y_2 = 0.300$$

$$x_3 = 0.020 \qquad y_3 = 0.352$$

$$x_4 = 0.000 \qquad y_4 = 0.363$$

$$m = \frac{y_4}{x_1} = \frac{1}{1.93} = \frac{0.363}{0.700} = 0.518.$$

To calculate I_1 (Figure 13)

$$c_1 = x_2$$

$$\frac{c_2}{y_2 - y} = \frac{1}{m}$$

thus,

$$c_2 = \frac{y_2}{m} - \frac{y}{m}$$

hence,

$$I_1 = \int_0^{y_2} \frac{c_1}{1-c_2} dy = m \int_0^{y_2} \frac{x_2}{1 - \frac{y_2}{m} + \frac{y}{m}} d\left(\frac{y}{m}\right)$$

$$= m\, x_2 \ln\left(1 - \frac{y_2}{m} + \frac{y}{m}\right) \Bigg]_0^{y_2} = m\, x_2 \ln \frac{1 - \frac{y_2}{m} + \frac{y_2}{m}}{1 - \frac{y_2}{m}}$$

$$= m\, x_2 \ln \frac{1}{1 - \frac{y_2}{m}} = \frac{0.120}{1.93} \ln \frac{1}{1 - 0.300 \times 1.93}$$

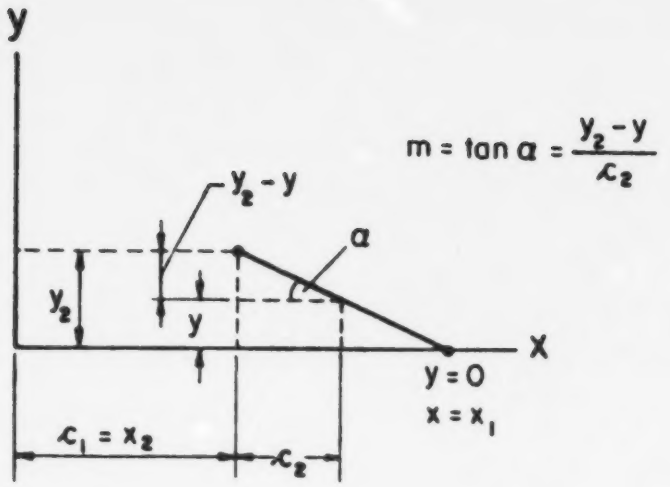

Fig. 13 — Schematic illustration of calculation techniques

hence,

$$I_1 = 0.0538.$$

To calculate I_2 (Figure 14)

$$c_1 = x_2$$

$$\frac{c_1 - x_3}{y_3 - y} = \frac{1}{m}$$

hence,

$$c_1 = \frac{y_3}{m} - \frac{y}{m} + x_3$$

with $c_2 = 0$, and $x_3 = 0.02$

thus,

$$I_2 = \int_{y_2}^{y_3} \frac{c_1}{1 - c_2} dy = \int_{y_2}^{y_3} \left(\frac{y_3}{m} - \frac{y}{m} + 0.02 \right) dy$$

$$= \frac{y_3}{m}(y_3 - y_2) - \frac{1}{2m}\left(y_3^2 - y_2^2\right) + 0.02(y_3 - y_2)$$

$$= \frac{1}{m}\left[y_3(y_3 - y_2) - \frac{1}{2}(y_3 - y_2)(y_3 + y_2) \right] + 0.02(y_3 - y_2)$$

$$= 1.93\left[0.352 \times 0.052 - \frac{1}{2} \times 0.052 \times 0.652 \right] + 0.02 \times 0.052$$

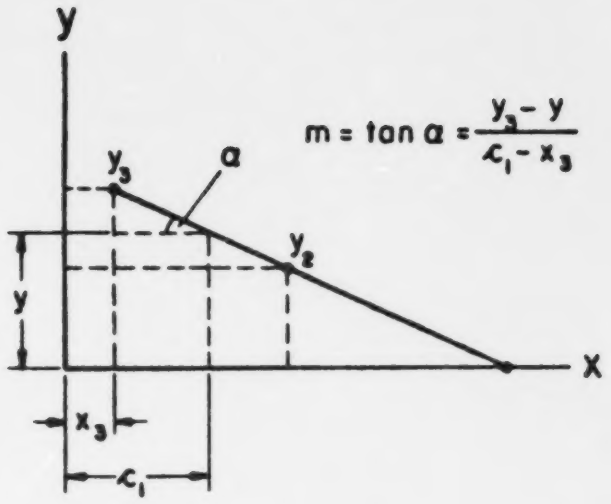

Fig. 14 — Schematic illustration of calculation techniques

hence,

$$I_2 = 0.0026 + 0.00104 = 0.00365.$$

To calculate I_3

$$c_1 = x_3, \qquad \text{and} \qquad c_2 = 0$$

$$I_3 = \int_{y_3}^{1} c_1 \, dy = \int_{y_3}^{1} x_3 \, dy =$$

$$= x_3 \, (1 - y_3) = 0.020 \times (1 - 0.352)$$

hence,

$$I_3 = 0.0130$$

and,

$$I = I_1 + I_2 + I_3 = 0.0538 + (0.0026 + 0.00104) + 0.0130 = 0.07044$$

given that:
$g = 9.81 \ (m/s^2)$; $s = 2.65$ (for gravel); $\rho_f = 1000 \ (kg/m^3)$; and $D = 0.263 \ (m)$

$$P_2 = g(s-1)\rho_f \, D \, I = 9.81 \times 1.65 \times 1000 \times 0.263 \times 0.07044 = 299.87 \ Pa.$$

This value is to be compared with $\Delta P_C = 300 \ Pa$ in Figure (11) of Shook et al. (1982a).

To calculate I with reference to Figure (12) of Shook et al. (1982a), we use

$$x_1 = 0.800 \qquad y_1 = 0.000$$

$$x_2 = 0.266 \qquad y_2 = 0.380$$

$$x_3 = 0.172 \qquad y_3 = 0.450$$

$$x_4 = 0.000 \qquad y_4 = 0.570$$

$$m = \frac{y_4}{x_1} = \frac{1}{1.40}$$

hence,

$$I_1 = m\, x_2 \ln \frac{1}{1 - \frac{y_2}{m}} = \frac{0.266}{1.40} \ln \frac{1}{1 - 0.380 \times 1.40} = 0.1443$$

$$I_2 = \frac{1}{m}\left[y_3(y_3 - y_2) - \frac{1}{2}(y_3 - y_2)(y_3 + y_2) \right] + x_3(y_3 - y_2)$$

$$= 1.40\left[0.450 \times 0.070 - \frac{1}{2} \times 0.070 \times 0.830\right] + 0.172 \times 0.07 = 0.01544$$

$$I_3 = x_3(1 - y_3) = 0.172 \times (1 - 0.450) = 0.0946$$

and,

$$I = 0.1443 + 0.0154 + 0.0946 = 0.25434.$$

Thus,

$$P_2 = 1082.74\ Pa.$$

This value is to be compared with $\Delta P_C = 1050\ Pa$ in Figure (12) of Shook et al. (1982a).

Buoyancy formulae

As shown earlier in this section, the density of the slurry is

$$\rho_{sl} = \rho_f \left[1 + \frac{(s-1)c_1}{1 - c_2}\right].$$

By assumption, contact particles have no dynamic support by the fluid. The submerged weight (per unit volume) of each contact particle in the slurry (containing only

suspended solids), is $g(\rho_s - \rho_{sl})$, where

$$\rho_s - \rho_{sl} = \rho_s - \rho_f\left[1 + \frac{(s-1)c_1}{1-c_2}\right] = (\rho_s - \rho_f) - \rho_f\frac{(s-1)c_1}{(1-c_2)}$$

$$= \rho_f(s-1) - \rho_f\left[\frac{c_1(s-1)}{1-c_2}\right] = \rho_f\left[(s-1) - \frac{(s-1)c_1}{1-c_2}\right]$$

$$= \rho_f(s-1)\left[1 - \frac{c_1}{1-c_2}\right] = \rho_f(s-1)\frac{1-c_1-c_2}{1-c_2}.$$

The total submerged weight of the entire contact component, per unit volume of the whole mixture (hence use of c_2), is

$$g\,c_2\,(\rho_s - \rho_{sl}) = g\,c_2\,\rho_f\,(s-1)\left[\frac{1-c_1-c_2}{1-c_2}\right].$$

If σ_s is the vertical stress exerted by the contact component, then

$$\frac{\partial\,\sigma_s}{\partial\,y} = g\,c_2\,\rho_f\,(s-1)\left[\frac{1-c_1-c_2}{1-c_2}\right] = g\,c_2\,(\rho_s - \rho_f)\left[\frac{1-c_1-c_2}{1-c_2}\right].$$

Hence, the foregoing expression agrees with Equation (B7) of Shook (1981, p. 22), and with Equation (14) of Shook et al. (1982a, p. 15). These authors give a more sophisticated derivation of the expression for $\frac{\partial\sigma_s}{\partial y}$, based on separated equations for the equilibrium of forces in the vertical direction, for the fluid, the suspended particles and the contact-load particles. The solution of these equations is presented in a manner somewhat different from that used by Shook (1981), in that it is derived more directly from the equations of the problem. The equilibrium equations are as follows (we omit the subscript y that indicated the vertical direction):

Vertical forces on the fluid:

$$0 = -\frac{\partial P}{\partial y} + \rho_f\,g + f_{fs} \qquad Eq\ 78$$

Vertical forces on the suspended particles:

$$0 = -\frac{\partial P}{\partial y} + \rho_s\,g + f_{sf} \qquad Eq\ 79$$

Vertical forces on the contact-load particles:

$$0 = -\frac{\partial P}{\partial y} + \rho_s\,g + f_{sw} \qquad Eq\ 80$$

where:

f_{fs} is the mean vertical force per unit volume of fluid, exerted on the fluid by the suspended particles,

f_{sf} is the mean vertical force per unit volume of solid material, exerted on the suspended particles by the fluid,

f_{sw} is the mean vertical force per unit volume of solid material, exerted on the contact-load particles by the pipe wall and transmitted only by other contact-load particles.

A further equation is derived from the balance of forces between the suspended load and the fluid, i.e.,

$$c_1 f_{sf} + (1 - c_1 - c_2) f_{fs} = 0 \qquad Eq\ 81$$

The contact-load particles do not contribute to this equation, except through the volume they occupy per unit volume of the mixture. From Equations 78 and 79, we obtain

$$f_{fs} - f_{sf} = (\rho_s - \rho_f) g. \qquad Eq\ 82$$

Then, by eliminating f_{sf}, using Equations 81 and 82, we obtain

$$f_{fs} = \frac{c_1}{1 - c_2} (\rho_s - \rho_f) g. \qquad Eq\ 83$$

Using Equation 82,

$$f_{sf} = f_{fs} - (\rho_s - \rho_f) g = -\frac{1 - c_1 - c_2}{1 - c_2} (\rho_s - \rho_f) g. \qquad Eq\ 84$$

Then, from Equations 79 and 80,

$$f_{sw} = f_{sf} = -\frac{1 - c_1 - c_2}{1 - c_2} (\rho_s - \rho_f) g. \qquad Eq\ 85$$

By the symmetry of the problem we can write, for points on the vertical diameter of the pipe,

$$\frac{\partial \sigma_y}{\partial y} = -c_2 f_{sw},$$

where σ_y is the vertical stress transmitted solely by the contact-load particles and supported solely by the wall of the pipe. Thus,

$$\frac{\partial \sigma_y}{\partial y} = -\frac{(\rho_s - \rho_f) g c_2 (1 - c_1 - c_2)}{(1 - c_2)}.$$

Similarly the vertical pressure gradient in the fluid is, from Equation 78,

$$\frac{\partial P}{\partial y} = \rho_f g + f_{fs} = g \frac{c_1 \rho_s + (1 - c_1 - c_2) \rho_f}{(1 - c_2)}.$$

Discussion

The foregoing theory seems to rely on an assignment of particles permanently to either contact load or suspended load. Its application also requires a knowledge of the distribution of c_1 and c_2 according to horizontal levels in the pipe. Shook (1983, pp. 7, 8) indicates that the purpose of the variable-composition distribution used in Shook (1981) and Shook et al. (1982a) was simply to show that the model used could produce actual measured head losses when using the actual total concentration distributions obtained by laboratory measurements. It was obviously an exploratory exercise, with interesting results. But the model suffers from difficulties that will be discussed below.

In the literature relating to the Wilson model discussions of contact- and suspended-loads usually refer to them as if they consist of separate and distinct populations of particles. In other words, each particle in the suspended-load would remain in that category, and similarly for the contact-load. If the two populations consisted of particles with radically different sizes, that apparent assumption would be easy to accept. But if all particles belong to a closely sized set, the assumption would be hard to accept.

It is not clear whether the assumption is tacitly accepted generally. If not, the terms *suspended-particle* and *contact-load particle* can only be justified by averaging effects that are not explained.

When the particles are closely graded in size it is more reasonable to assume that, on the average, they all are semi-suspended if suspension exists, rather than to assign some particles to full suspension and the remainder to contact-load.

In the terminology of Shook et al., if all particles were to have the same size the Equations 78, 79 and 80 would reduce to the following two equations applying to fluid and to particles, respectively.

$$0 = -\frac{\partial P}{\partial y} + \rho_f g + f_{fs} \qquad Eq\ 86$$

$$0 = -\frac{\partial P}{\partial y} + \rho_s g + f_{sf} + f_{sw}. \qquad Eq\ 87$$

If the local concentration of particles is c, the balance of forces between particles and fluid is

$$c f_{sf} + (1 - c) f_{fs} = 0 \qquad Eq\ 88$$

Equations 86 to 88 can be solved similarly to Equations 78 to 81, as follows:

From Equations 86 and 87,

$$f_{fs} - f_{sf} - f_{sw} = (\rho_s - \rho_f) g. \qquad E_{\iota}\ 89$$

From Equations 88 and 89,

$$f_{fs} = c\,[f_{sw} + (\rho_s - \rho_f)\,g]. \qquad Eq\ 90$$

Then, using Equations 89 and 90,

$$f_{sf} = f_{fs} - f_{sw} - (\rho_s - \rho_f)\,g = -(1-c)\,[f_{sw} + (\rho_s - \rho_f)\,g]. \qquad Eq\ 91$$

If all the particles are fully suspended, $f_{sw} = 0$, and

$$f_{sf} = -(1-c)(\rho_s - \rho_f)\,g$$

which is what Equation 84 reduces to when $c_2 = 0$.

If all the particles are unaffected by dynamic suspensive fluid forces, $f_{sf} = 0$, and

$$f_{sw} = -(\rho_s - \rho_f)\,g$$

which is what Equation 84 reduces to when $c_1 = 0$.

It is clear that when particles are in an intermediate state of suspension, f_{sf} and f_{sw} cannot be determined separately from Equations 89 to 91. It is therefore difficult, if not impossible, to incorporate semi-suspension into a two-level model.

The Wilson model includes only full suspension or full contact. It assigns particles of each size partly to suspension and partly to contact load, according to Equation 32 of this report. Particle size is involved, but after the separation has been made their diameters are irrelevant (in Wilson's original approach).

Shook et al. do not use the Wilson model in the work discussed here, when determining C_1 and C_2. They adjust the latter concentrations (using an arbitrary system of distribution) to attain harmony between the measured hydraulic gradients shown in their Figure (10) and the measured total-concentration distributions shown in Figures (11) and (12), using the same friction factor η_s, which is also adjusted to achieve the results shown.

But, except for the separation of C_1 and C_2, the authors use a two-level model similar to the Wilson model. The position of the upper boundary of the bed is not stated, but it seems likely that it was placed at the level where c_2, the local concentration of contact-load particles, vanishes. This would mean that the top level of the bed would consist largely of suspended particles. The transition from bed to slurry is gradual, but no modification to the interfacial friction factor is mentioned.

The authors report that shearing was observed in the high-concentration lower layer and they suggest that the sliding-bed hypothesis is not an essential feature of the two-layer

model. This adds to previous evidence (see p. 21) of the unrealistic nature of the Wilson model. The model used by Shook et al. must also be regarded as unrealistic.

The friction factor η, used in all models similar to the Wilson model, cannot be viewed as a simple factor of mechanical rubbing friction between particles and pipe wall. Similarly, application of the buoyancy calculations to a two-layer model cannot be regarded as being physically well founded, owing to the well-known extreme complexity of turbulence.

This is not to say that such exploratory work is pointless. Useful empirical formulae have been arrived at by such work. But the final test is whether a relatively simple, usable formula is found, whatever its anomalies are.

One anomaly of the Wilson model is that the basic equation (Wilson, May 1976)

$$\frac{C_{dc}}{C_d} = \left(\frac{V_t}{V}\right)^{\alpha}$$

uses delivered concentrations, C_d and C_{dc}, whereas the force balance equations must use in situ concentrations of solids. However, this equation was derived from data plotted in such a way that C_{dc} would represent, in Wilson's concept, the net resistance that the contact load exerts through the friction factor η. Thus, it is difficult to see how a correspondence could be found between the Wilson model and that used by Shook et al. There is no place in the former for a buoyancy effect that would subtract from the resistant effect of the contact load calculated by the model.

It is suggested here that buoyancy problems are so complex that buoyancy forces other than static should be omitted from the sliding bed. Particles in the bed should be considered as contact load, whatever their sizes. It is questionable whether any such particles, even small ones, can physically be in full suspension unless all are. Similarly, all particles above the bed should be considered in suspension, whatever their sizes. The model is not realistic, but its employment must be consistent.

The Wilson model can fit a series of runs with the same material and load value at different velocities, using a constant value of the friction factor η; but, because it requires different values of η for different load values, an empirical study should be made of η as a variable. As η does not represent a true physical friction factor, it need not be assumed to be constant. It is a function of the parameters of the flow. Yet, a prime incentive for the buoyancy calculations seems to have been to preserve the constancy of η.

An empirical study of η as a variable is suitable to the problem of improving the Wilson model (see the discussion commencing on p. 47). The results of such work might well allow the retention of the model's advantage in relative simplicity, while extending its area of application.

Problem of the "Effective" Density of Porous Particles

In general, ρ_s represents the density of particulate materials, but if the material is porous, an ambiguity arises.

Shriek et al. (1973) give a standard method for measuring the density of dry coal. The method clearly intends to determine the mass of a sample of coal from which water has been removed from the accessible pores. The volume of the sample is detemined in such a way that it represents the coal material only. It does not include the volume of the accessible pores.

The ratio of the foregoing quantities, i.e., mass divided by volume, gives the density of dry coal, which has been represented by ρ_s.

Because the density of water-saturated coal occurs in problems of hydraulic transport (see Shen et al., 1976, p. 5; Eyler et al., 1982, p. 4.4), and because ρ_s is generally used for the density of particles in this field, it is proposed that this symbol be used for the density of saturated coal, and that ρ_{ss} be used for dry coal, when the distinction must be made.

Now, let m represent the moisture concentration (by weight, per unit weight) in water-saturated coal. Then, to obtain the *wet volume* of saturated coal, we can write

$$\frac{1}{\rho_s} = \frac{1-m}{\rho_{ss}} + \frac{m}{\rho_f} \qquad \left(\frac{cc}{g}\right) \qquad Eq\ 92$$

(as on page 8 of Haas et al. (1980), for the case of coal with a dry density of 1.561 (g/cc), and a moisture content of 24.6%).

Definitions of the following quantities are required:

ρ_f = density of water,
C_w = concentration by weight of particulates in slurry,
W_s = weight of saturated particulates from a sample of slurry,
W_f = weight of free water from same sample of slurry, excluding moisture contained in particulates.

$C_w = \frac{W_s}{W_s + W_f}$ appears to be the normal definition of C_w, in accordance with the usage of the Saskatchewan Research Council (e.g., see Gillies et al. 1981, p. 15).

Thus,

$$C_w = \frac{1}{1 + (W_f/W_s)}$$

and,

$$\frac{W_f}{W_s} = \frac{1 - C_w}{C_w} \qquad Eq\ 93$$

Further,

C_v = concentration by volume of particulates in slurry,

$V_s = W_s/\rho_s$ = volume of particulates from the same sample,

$V_f = W_f/\rho_f$ = volume of free water from the same sample.

Thus,

$$C_v = \frac{V_s}{V_s + V_f} = \frac{(W_s/\rho_s)}{(W_s/\rho_s) + (W_f/\rho_f)} = \frac{1}{1 + (W_f/W_s)(\rho_s/\rho_f)}. \qquad Eq\ 94$$

Since C_v is a general term, it is suggested that this term should not be used for the concentration of a delivered slurry. C_d seems to be acceptable, as C_D, for the *drag coefficient*, is easily distinguishable. Analogously, C_t seems acceptable for the concentration of the slurry in transport (namely, for the in situ one, in the pipe).

The significance of certain data from the Slurry Pipeline Development Centre of the Saskatchewan Research Council requires clarification, with reference to several reports. Haas et al. (1980) state clearly that the density for water-saturated coal, ρ_s, is required for calculation of C_v from C_w, and for use in the determination of C_D (from the *terminal falling velocities*, w_t). Both densities, for dried and for water-saturated coal, are given for several coals (see Haas et al., 1980, p. 11, Table 2). In the related paper (Shook et al., 1981, p. 86, Table 1), however, only ρ_{ss} has been given for Sheerness and McIntyre coals. ρ_s was used for calculations, according to the text, but values of ρ_s were not given as an aid against misunderstanding.

In this context the following papers must also be mentioned, because they too leave the reader in doubt as to the proper use of water-saturated coal densities. The texts do not, in fact, appear to be consistent with tabulated data, and with some of the figures.

Shook (1980) and Gillies et al. (1981) deal with data for Judy Creek Coal, which is porous. Data for this coal are given as

ρ_{ss} = 1,727 (kg/m^3)

m = 0.235 (moisture concentration)

ρ_f = 998 $(T = 20°C)$

Then, from Equation 92, ρ_s = 1,473 (kg/m^3) = density of particulate material.

Now, consider the runs summarized on p. A45 of Gillies et al. (1981); for example,

C_w = 0.366 (concentration by weight), and

ρ_{sl} = 1,181,

where, ρ_{sl} represents the density of the slurry in transport.

Calculations give the following results:

$$\frac{W_f}{W_s} = \frac{1 - C_w}{C_w} = 1.732$$

from Equation 93, with $C_w = 0.366$.

Also, from Equation 94

$$C_t = \frac{1}{1 + (W_f/W_s)(\rho_s/\rho_f)} = \frac{1}{1 + 1.732(1.473/0.998)} = 0.281187$$

with, $\rho_{sl} = C_t\ \rho_s + (1 - C_t)\rho_f = 1,131.$

This does not agree with the tabulated value of 1,181, for ρ_{sl}.

When we use ρ_{ss}, in place of ρ_s, we find that

$$\rho_{sl} = 0.2502 \times 1.727 + (1 - 0.2502) \times 0.998 = 1.1804$$

i.e., a value more or less in agreement with the tabulated value of ρ_{sl}. We also find 0.250 for C_t, which is a value commonly quoted in association with $C_w = 0.366$, a standard condition for numerous runs. Clarification of the data for Judy Creek coal is needed.

Meagre information on the precise use of coal density is given in both relevant papers. No indication of the employment of ρ_s can be found in the data or the figures. Serious difficulties occur in Figure (3), p. 20, of Shook (1980), in which the top diagram includes a model fit and plots of gamma-ray gauge concentration measurements. Table (5), p. 22, of Gillies et al. (1981) gives the gamma-ray data, but lists $C_v = 0.250$, indicating that density measurements are not given with respect to water-saturated coal. Figure (77), p. 109, of Gillies et al. (1981) is similar to that of Shook.

It is recommended that ρ_s, the "effective" density of water-saturated coal, should be given explicitly in all cases, and that C_t should clearly be calculated by the use of ρ_s.

Problem of the Calibration of a Gamma-Ray Concentration Gauge

Gillies et al. (1981, p. 13) state that the absorption coefficient for coal could not be determined directly with sufficient accuracy. Consequently, a set of measurements made on a pipeline, run with a known constant volume fraction of coal, was used to determine the absorption coefficient for coal.

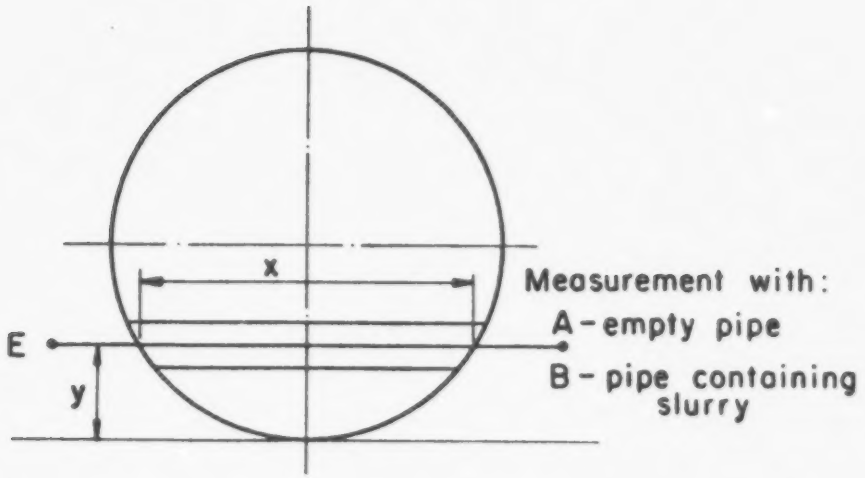

Fig. 15 — Schematic arrangement of test set-up

The following calculations verify our understanding of the process in question.

Let (see Figure 15)

E represent the gamma-ray source,

y be a known height of the beam, measured from the bottom of the pipe,

x be the length of path of the beam within the pipe,

A be the gamma-ray measurement, obtained when the pipe is empty,

B be the measurement under the same conditions as A, except that a run is in process with slurry in the pipe,

k_f =0.0847 be the known gamma-ray absorption coefficient for water,

k_s be the assumed coefficient for coal,

k_m be the measured coefficient, determined from measurements at a height y,

C be the mean in situ fraction of coal, by volume in a chord-slice, at height y.

By the exponential law for attenuation of the gamma-ray beam by the presence only of the slurry, we obtain the expression

$$\frac{B}{A} = e^{-k_m x}$$

and, therefore,

$$\ln\left(\frac{B}{A}\right) = -k_m\, x.$$

Knowing $\frac{B}{A}$ and x, k_m is calculated. But, by definition of the absorption coefficient, $k_m = C\, k_s + (1 - C)k_f$. Therefore, $C = \frac{k_m - k_f}{k_s - k_f}$, and C can thus be determined at each level of y.

Let the levels of y/D, tabulated in Table (5) of Gillies et al. (1981, p. 22), be called y_i', $i = 1, \ldots, 10$ (Figure 16). Each y_i' represents a horizontal slice (i) of the pipe cross section. The boundaries of the slices, distinguished as Y/D, will be called Y_i', $i = 0, \ldots, 10$. Slice (1) is bounded by $Y_0' = 0.000$, and $Y_1' = 0.106$. Slice (10) is bounded by 0.906 and 1.000. All slices ($i = 2$ to 9) have boundaries $Y_{i-1}' = y_i' - 0.050$ and $Y_i' = y_i' + 0.050$.

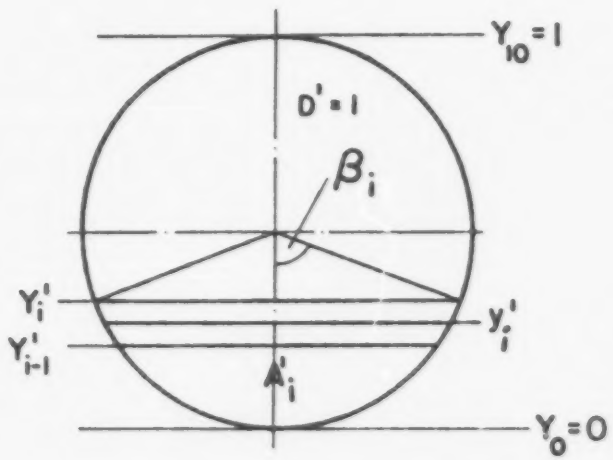

Fig. 16 — Schematic illustration of calculation techniques

Also:

A_i' is the fractional area of the sector bounded by Y_i' and Y_{i-1}',

$y = (D/2)(1 - \cos\beta)$,

$y' = (y/D) = \frac{1}{2}(1 - \cos\beta)$,

$\cos\beta = 1 - 2y'$,

$A = (D^2/4)(\beta - \sin\beta\cos\beta)$ (see Figure 17), and

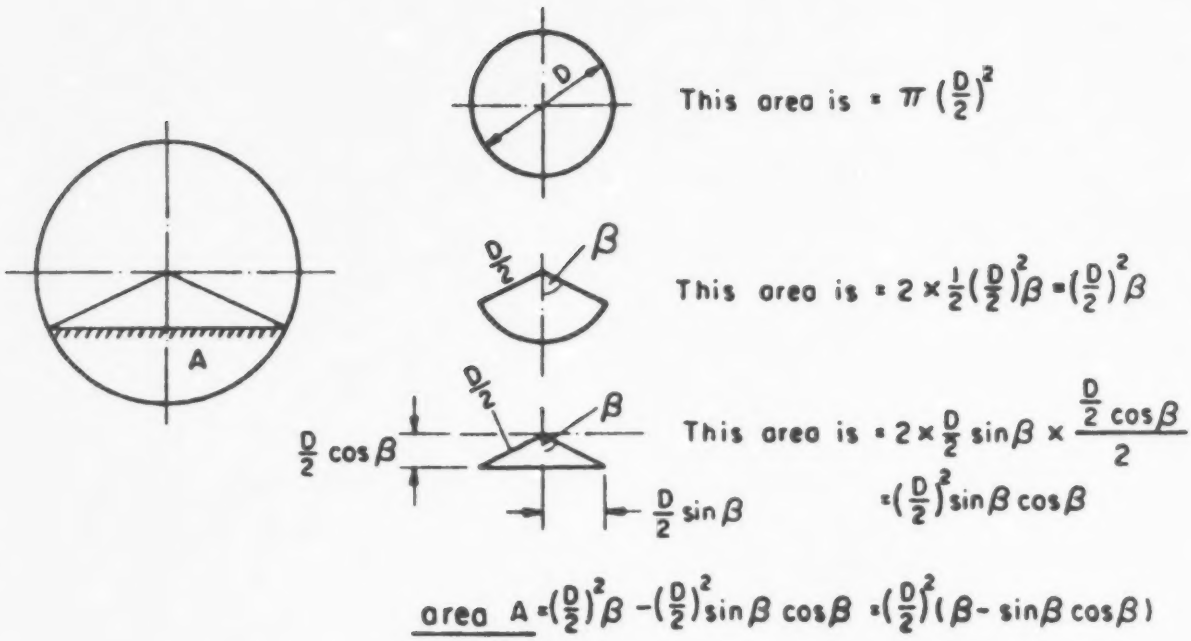

Fig. 17 — Schematic illustration of calculation of area A

$$A' = A/(\pi D^2/4) = \tfrac{1}{\pi}(\beta - \sin\beta\cos\beta).$$

The areas of the slices are given by

$$\Delta_i = A'_i - A'_{i-1}.$$

For each slice the measured solids volume fraction, tabulated in Table (5) of Gillies et al. (1981, p. 22), is given by

$$C_i = \frac{k_{mi} - k_f}{k_s - k_f}.$$

The product $\Delta_i \times C_i$ gives the solids volume fraction contributing, by slice (i), to the total solids volume fraction of the pipe, C,

$$C = \sum_1^{10} \Delta_i C_i = \frac{1}{k_s - k_f} \sum_1^{10} (k_{mi} - k_f)\Delta_i = \frac{1}{k_s - k_f} X.$$

Results of the calculations, verifying C for the two runs of Gillies et al. (1981), are shown in Tables 10 and 11.

TABLE 10
Concentration profile measurements with a gamma-ray instrument

i	$Y_i' = \frac{Y_i}{D}$	$A_i' = \frac{\beta - \sin\beta\cos\beta}{\pi}$	$\Delta i = A_i' - A_{i-1}'$	$y_i' = y_i/D$ (*)
0	0.000	0.0000		
			0.0567	
1	0.106	0.0567		0.056
			0.0918	
2	0.206	0.1485		0.156
			0.1108	
3	0.306	0.2593		0.256
			0.1217	
4	0.406	0.3810		0.356
			0.1266	
5	0.506	0.5076		0.456
			0.1264	
6	0.606	0.6340		0.556
			0.1207	
7	0.706	0.7547		0.656
			0.1090	
8	0.806	0.8637		0.756
			0.0888	
9	0.906	0.9525		0.856
			0.0475	
10	1.000	1.0000		0.956

(*) as in Table (5), of Gillies et al. (1981, p. 22).

The two values of C found, namely, $C_1 = 0.249$ and $C_2 = 0.251$, confirm the manner of calibration of k_s, since they are consistent with the value, $C = 0.250$, given by Gillies et al. (1981). This value was not assumed in the foregoing calculations. Thus, we can determine the experimentally measured sum $X = \sum_1^{10}(k_{mi} - k_f)\,\Delta_i$ for each run as follows:

Run 1

$$X_1 = (k_s - k_f)C = (0.141 - 0.0847) \times 0.2491 = 0.01402.$$

In a previous report-section (see p. 70) it was shown that the density of water-saturated coal seems to lead to a different value for these runs, namely to $C_t = C' = 0.281187$, i.e., $C_1' = C_2' = 0.281$.

To recalculate k_s, we would repeat the calculation with the new C' value, i.e.,

$$(k_s' - k_f) = \frac{X_1}{C_1'} = \frac{0.01402}{0.281187} = 0.04986$$

hence,

$$k_s' = \frac{X_1}{C_1'} + k_f = 0.04986 + 0.0847 = 0.1346.$$

TABLE 11

Concentration profile measurement calculations

	Run 1•		Run 2ℵ	
i	$(C_i)^{(*)}$	$\Delta_i.C_i$	$(C_i)^{(*)}$	$\Delta_i.C_i$
1	0.568	0.03220	0.554	0.03141
2	0.483	0.04436	0.441	0.04050
3	0.377	0.04178	0.329	0.03646
4	0.274	0.03334	0.257	0.03127
5	0.206	0.02608	0.218	0.02760
6	0.165	0.02084	0.197	0.02488
7	0.147	0.01775	0.173	0.02088
8	0.140	0.01526	0.166	0.01810
9	0.126	0.01119	0.142	0.01261
10	0.133	0.00632	0.154	0.00732
Hence		$C_1 = 0.24912$	and $C_2 = 0.25103$	

• $V = 3.41$ m/s; $k_s = 0.141$; $k_f = 0.0847$ (*).

ℵ $V = 4.06$ m/s; $k_s = 0.136$; $k_f = 0.0847$ (*).

(*) as in Table 5 of Gillies et al. (1981, p. 22).

Similarly, for Run 2:

$$X_2 = (0.136 - 0.0847) \times 0.2510 = 0.01288$$

and,

$$(k'_s - k_f) = \frac{X_2}{C'_2} = \frac{0.01288}{0.281} = 0.0458$$

hence,

$$k'_s = 0.1305.$$

Thus,

$$C'_i = \frac{k_{mi} - k_f}{k'_s - k_f}$$

analogously to

$$C_i = \frac{k_{mi} - k_f}{k_s - k_f}.$$

Then,

$$\frac{C'_i}{C_i} = \frac{k_s - k_f}{k'_s - k_f} = \frac{C'}{C}.$$

TABLE 12

Amended concentration profile, allowing for "effective" density correction

i	Run 1 — C_i'	Run 2 — C_i'
1	0.638	0.623
2	0.543	0.496
3	0.424	0.370
4	0.308	0.289
5	0.232	0.245
6	0.185	0.221
7	0.165	0.194
8	0.157	0.186
9	0.142	0.160
10	0.149	0.173

Consequently, the solids volume fractions given in Table (5), of Gillies et al. (1981, p. 22), should be multiplied by $(C'/C) = (0.281/0.250) = 1.124$, to give the results shown in Table 12. These appear to be the results that would have been calculated by Gillies et al. (1981) if the density of water-saturated coal, ρ_s, had been used, not that of dry coal, ρ_{ss}, (in the notation of this report).

Comments on the Dimensionless Groups used by Gillies et al.

Gillies et al. (1985, Section 3.2) have developed an empirical expression for $\frac{C_f}{C_r}$ (or $\frac{C_f}{C_b}$ in the notation of this report) in terms of the dimensionless groups:

$$\frac{V^2}{g\,d},\ \frac{d}{D},\ \frac{\rho_s - \rho_f}{\rho_f}, \quad \text{and,}$$

$$A_r = 4\,g\,d^3\,\rho_f\,(\rho_s - \rho_f)/3\,\mu_f^2 = \frac{4}{3}\,\frac{g\,(\rho_s - \rho_f)\,d^3}{\rho_f\,\nu_f^2} = \frac{4}{3}\,\frac{\gamma_s\,d^3}{\rho_f\,\nu_f^2} = \left[\frac{4}{3}\,\Xi\right]^{\P}$$

For reasons described in detail by Yalin (1977) and outlined in Appendix B of this report, as well as in the next report section, it would be advantageous, in a particular

¶ See Table (B-3), in Appendix B, and also see Appendix C.

problem, to employ consistent dimensionless groups that are the dimensionless variables based on a single set of basic parameters.

The dimensionless variable Ξ is based on the set of basic parameters d, ρ_f, and γ_s, and it is of interest to express all the groups employed by Gillies et al. (1985) in terms of the same basic parameters. The related dimensionless variables are akin to those shown in Table (B-3), in Appendix B.

Because the mean velocity V is employed by Gillies et al. (1985), V replaces the friction velocity v_*. This is quite legitimate because V and v_* have the same dimensions and can replace each other as a characteristic velocity of the flow. The dimensionless variable $Y = \frac{\rho_f v_*^2}{\gamma_s d}$ is replaced by $Y' = \frac{\rho_f V^2}{\gamma_s d}$.

Then we can write

$$\frac{V^2}{g\,d} = \frac{\rho_f V^2}{g\,(\rho_s - \rho_f)\,d} \times \frac{\rho_s - \rho_f}{\rho_f} = \frac{\rho_f V^2}{\gamma_s d}\left(\frac{\rho_s}{\rho_f} - 1\right) = Y'\,(W-1)$$

$$\frac{d}{D} = Z^{-1}$$

and,

$$\frac{\rho_s - \rho_f}{\rho_f} = W - 1.$$

It follows that

$$\begin{aligned} & A_r^{k_2}\left(\frac{V^2}{g\,d}\right)^{k_3}\left(\frac{d}{D}\right)^{k_4}\left(\frac{\rho_s-\rho_f}{\rho_f}\right)^{k_5} \\ &= \left(\frac{4}{3}\Xi\right)^{k_2}\left[Y' \times (W-1)\right]^{k_3} Z^{-k_4}\left(W-1\right)^{k_5} \\ &= \left(\frac{4}{3}\Xi\right)^{k_2}\left(Y'\right)^{k_3} Z^{-k_4}\left(W-1\right)^{k_3+k_5}. \end{aligned}$$

This suggests that it would be better to write

$$\frac{C_c}{C_r} = exp\left[\kappa_1\left(\Xi\right)^{\kappa_2}\left(Y'\right)^{\kappa_3}\left(Z\right)^{\kappa_4}\left(W\right)^{\kappa_5}\right]$$

since Y' represents the total effect of gravity on $(\rho_s - \rho_f)$, and W represents the total inertial effects of ρ_s, i.e., the gravity and the inertial effects are separated.

If the effects of inertial forces of solid particles were negligible, κ_5 would be close to zero, and the variable W could be dropped. No such conclusion could be drawn in

connection with the expression used by Gillies et al. (1985), because the term in $\frac{\rho_s - \rho_f}{\rho_f}$ would be affected by both gravitational and inertial forces.

Inclusion of an additional term, in $X_5 = C$ (see Table (B-3)), would be of interest(*). $C = C_d$ or $C = C_t$ could be used. Of course, two further dimensionless variables, $X_6 = k/d$ and $X_7 = \eta$, appear in Table (B-3). Terms in those variables should, strictly speaking, be included, if data warranted it.

Discussion of the Effects of Particle Density, ρ_s, and of Sliding Friction, η

In Appendix B, slurry flows are compared only in case of fixed particle parameters, except for a variation of diameter d, under the restriction that D/d be constant. It is of interest also to make comparisons between flows with differing particle densities. It will be appreciated that, in connection with natural materials, a variety of changes in parameters may be associated with a change in density. The possible consequences should be considered.

Yalin (1977, p. 69) shows that ρ_s(†) does not influence any steady particle velocity involved in a hydraulic flow (see also Appendix C). The parameter ρ_s only appears directly (i.e., through the variable $W = \rho_s/\rho$) in expressions for particle acceleration. According to Yalin,

> "it follows that the ratio W can be important only with regard to the properties associated with the 'ballistics' of an indvidual grain. Usually, in engineering practice, one is much more interested in the properties of the grain motion en masse and, with regard to these properties, the ratio W appears to be the least important variable."

The other variables considered by Yalin in connection with sediment transport are only X, Y, Z, or Ξ, Y, Z, or equivalent.

At first sight, slurry flow seems to be a case where W should have little importance. It would appear to be safe to ignore the variable W, especially in comparisons of similar solids. However, serious difficulties can be foreseen, except under special circumstances.

The mechanisms of sediment transport in open flows considered by Yalin and others do not include mechanical friction as an independent parameter. Losses of energy are

(*) See the discussion of Babcock's results, in the section of this report entitled "Delivered Concentration, C_d, as a Parameter" (p. 11).
(†) ρ_s = particle density $kg/m^3 \rightarrow ML^{-3}$.

assigned, principally, to hydraulic processes; mechanical friction losses, if considered at all, are absorbed into empirical constants. The link between ρ_s and the "hydraulic processes" mentioned, is the steady attraction of gravity, acting through $\gamma_s^{(**)}$, that tends to make particles settle, whereas turbulence, or lift effects, tend to annul such settling.

Newitt et al. (1955) mention mechanical friction as a cause of energy loss in a slurry flow in a pipe when a moving bed exists, but again the friction factor is absorbed into an empirical constant. The combination of parameters $\eta(\rho_s - \rho)\ g = \eta\ \gamma_s$ occurs in the constant.

The Wilson model explicitly introduces mechanical friction and the expression for resistance due to a moving contact-load again includes $\eta\ \gamma_s$.

If the total effect of ρ_s acts through γ_s, then η will be a component of any empirical function of γ_s. This parameter γ_s will remain a valid one, as stated by Yalin, when only one particulate solid is concerned (since η is then a constant multiplier of γ_s). If different solids are concerned, η may vary, with or without variation of γ_s. We see that only the product $\eta\ \gamma_s$ can, in general, be considered as a single valid parameter, as far as the effect of contact load is concerned. If γ_s is used on its own, an unaccountable scatter in the correlation of flow properties will occur, due to any differences in η.

In pipe flow we can also expect a second complication, that probably has little importance in sedimentary flows in open channels. A brief summary of abrasion in pipes (Bain and Bonnington, 1970, p. 131) indicates clearly that even a fine abrasive solid carried in homogeneous suspension causes wear. Some mechanical friction loss must be associated with wear, due to rubbing on the pipe wall.

Further, if the particles are thrown against the wall by turbulence, acceleration is involved. As shown by Yalin, the parameter ρ_s is concerned in this case, not γ_s. Consequently, the product $\eta\ \rho_s$ must be concerned with this source of energy loss.

Summarizing, we find three distinct mechanisms for energy loss involving particle density:

- hydraulic losses depending on γ_s, the principal loss recognized by Yalin (1977);
- mechanical friction associated with sliding contact-load particles, dependent on $\eta\ \gamma_s$;
- mechanical friction associated with hydraulically supported particles, dependent on $\eta\ \rho_s$.

(**) $\gamma_s = (\rho_s - \rho)\ g$ = specific net weight of a particle submerged in fluid
$\frac{m}{s^2}\frac{kg}{m^3} = \frac{kg}{s^2\ m^2} \rightarrow M\ L^{-2}\ T^{-2}$.

Because complex sharing of energy loss is not understood theoretically, empirical studies will have to be made with these points in mind. It will probably be impossible to isolate all effects. In the simplest situation a conditional test of their significance can be made thus: let us assume that two different particulate solids (e.g., coals) are similar in all respects except in density. Also, let us assume, initially, that W has no effect on slurry flows, so that the whole effect of ρ_s is reflected in γ_s.

Now, let flow tests be carried out under conditions of geometric similarity in the same pipe, with the same C values, and with the same characteristic particle diameter, i.e., $D_1 = D_2$, $C_1 = C_2$, and $d_1 = d_2$. Corresponding values of i_1 and of V_1 are then to be measured in a series of runs with the first material, of density ρ_{s1}, and similarly values of i_2 and V_2 with the second material, of density ρ_{s2}.

As a general rule, a parameter whose effect is to be tested should not be selected as a basic parameter, because basic parameters may appear in several variables. As an example from Table (B-3) of Appendix B, γ_s appears in Ξ as well as in Y, and also in the related variable $\overline{\overline{\Pi}}_V$.

Parameters, other than basic, each appear in one variable only, the variable that *represents*, or *reflects*, its effect. Note that in Table (B-2) of Appendix B, γ_s appears only in Y, because Y represents γ_s. We, therefore, choose the dimensionless variables based on the basic parameters d, ρ, and v_s (see Table (B-2)) for analyzing the results of the present series of tests.

By the previously stated hypothesis, the two flows should be similar when $Y_1 = Y_2$ (with $Z_1 = Z_2$, and $C_1 = C_2$). Therefore, $\Pi_{V1} = \Pi_{V2}$ also.

From the first relation,

$$Y_1 = \frac{\rho v_{s1}^2}{\gamma_{s1} d_1} = \frac{\rho v_{s2}^2}{\gamma_{s2} d_2} = Y_2.$$

Since $d_1 = d_2$,

$$\frac{v_{s1}^2}{v_{s2}^2} = \frac{\gamma_{s1}}{\gamma_{s2}}.$$

Since $\frac{d_1}{D_1} = \frac{d_2}{D_2}$, $D_1 = D_2$; and since $v_{s1} = \sqrt{i_1\ g\ D_1/4}$, and $v_{s2} = \sqrt{i_2\ g\ D_2/4}$,

$$\frac{i_1}{i_2} = \frac{\gamma_{s1}}{\gamma_{s2}} = \frac{g\ (\rho_{s1} - \rho)}{g\ (\rho_{s2} - \rho)} = \frac{(s_1 - 1)}{(s_2 - 1)}$$

where, $s_1 = \rho_{s1}/\rho$, and $s_2 = \rho_{s2}/\rho$, but $s_1 \neq s_2$, because $\rho_{s1} \neq \rho_{s2}$.

From the second relation above (i.e., from $\Pi_{V1} = \Pi_{V2}$),

$$\frac{V_1}{v_{s1}} = \frac{V_2}{v_{s2}}$$

hence,

$$\frac{V_1}{V_2} = \frac{v_{*1}}{v_{*2}} = \sqrt{\frac{i_1}{i_2}} = \sqrt{\frac{(s_1 - 1)}{(s_2 - 1)}}.$$

If the data of both series of runs fulfil the latter relations, both sets of points will fall on a single line, if plotted as in Figure (B-2) of Appendix B. If, however, the latter relations are not fulfilled, two different lines will result. In either case no unambiguous interpretation can be made, unless $\eta_1 = \eta_2$.

If $\eta_1 = \eta_2$, a single line indicates that ρ_s has no direct effect upon the flow. In this case it is the other parameters, in particular γ_s, that account for the results. In these special circumstances, a series of runs with materials of different densities could be similarly correlated, through the use of either Y and Π_V (Figure (B-2) of Appendix B), or Y and $\overline{\overline{\Pi}}_V$ (Figure (B-4) of Appendix B), even if pipe diameters D_1 and D_2 were different‡.

If $\eta_1 = \eta_2 = \eta$, and two distinct lines occur, this must indicate the existence of a significant friction loss through contacts of suspended particles with the wall. The difference between the lines depends on the difference between $\eta\ \rho_{s1}$, and $\eta\ \rho_{s2}$.

A correlation, such as that shown in Figures (B-2) or (B-4) of Appendix B, requires that differences due to the effects of X, W, X_6, X_7, and particle shape be relatively unimportant. Because two natural materials differing in density may differ in other parameters as well, say in shape or mechanical friction or both, it is likely that the actual effects of these variables will lead to a scatter of points about the line.

Problem of Defining Velocity V, and of Interpreting Equation (A.18) of Eyler et al.

Shook (1983, p. 8) remarks on an apparent error in Equation (A.18) of Eyler et al. (1982, p. A.15), and suggests (see his Appendix (2)), that it appears to arise from the evaluation of V_a/V.

It is shown here that the discrepancy is caused by different definitions for the symbol V, used by Shook and by Eyler et al. Other differences in notation are only minor. The notation of Eyler et al. (1982) is used, except that $V_{(Shook)}$ denotes V as defined by Shook.

Shook (1983, p. *ii*) defines $V_{(Shook)}$ as the *average velocity*, indicating the average velocity of fluid and of solids across the whole pipe. This is equivalent to $V_{m(Eyler)}$, defined by Eyler et al. (1982, p. *xix*) as *mixture velocity*, i.e., $V_{(Shook)} = V_{m(Eyler)}$.

‡ Because conditions for a single line are fulfilled; we do not, actually, have identical points from the pairs of runs, but they do fall on the same line.

Thus, $A \times V_{m(Eyler)} = A_a V_a + A_b V_b$, which can be reduced to

$$(1-a)\frac{V_b}{V_{m(Eyler)}} = 1 - a\frac{V_a}{V_{m(Eyler)}}. \qquad Eq\ 95$$

The volumetric rate of delivery of solids* can be given by two equivalent expressions, i.e.,

$$C_d\, A\, V_{m(Eyler)} = C_b\, A_b\, V_b$$

hence,

$$C_R = \frac{C_d}{C_b} = \frac{A_b}{A} \times \frac{V_b}{V_m} = (1-a)\frac{V_b}{V_{m(Eyler)}} = (1-a)\frac{V_b}{V_{(Shook)}}$$

or,

$$C_R = (1-a)\frac{V_b}{V_{m(Eyler)}} = 1 - a\frac{V_a}{V_{m(Eyler)}} \qquad \text{(from Equation 95).}$$

Hence,

$$\frac{V_a}{V_{m(Eyler)}} = \frac{1-C_R}{a} \qquad Eq\ 96$$

as given by Shook (1983, p. 33).

Eyler et al. (1982, p. *xix*) define** $V_{(Eyler)}$ as *equivalent throughput velocity.* A discussion, commencing on their p. 5.6 makes it clear that $V_{(Eyler)}$ represents a velocity relevant to the fluid component only, if delivered through the whole pipe cross section.

In case of a *fixed bed*, $V_b = 0$, and $V_{(Eyler)} = a\, V_a$.

For the case of a *moving bed*, an expression for $V_{(Eyler)}$ is not given, and it is simply said (bottom of their p. 5.9), that " a more complex expression similar to Equation (5.7) results." As their Equation (5.7) gives an expression for $V_{m(Eyler)}$, there is a strong indication that $V_{(Eyler)}$ again refers to output of fluid only.

An exression for $V_{(Eyler)}$ is found as follows. It is *assumed that there are no suspended solids* in the above-bed region, and that the volumetric concentration of solids in the bed region is C_b, in accordance with the Wilson model.

The volumetric rate of delivery of fluid only can be given by two equivalent expressions, i.e.,

$$A\, V_{(Eyler)} = A_a\, V_a + A_b\, V_b\, (1 - C_b)$$

(assuming that the fluid velocity is V_b).

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* Assuming full stratification, i.e., that no solids travel in suspension.
** Note also their definitions of j_0 and of j_{0s}.

Hence,

$$V_{(Eyler)} = a\,V_a + (1-a)\,V_b\,(1-C_b) = a\,V_a\left[1 + \frac{1-a}{a}\,V_R\,(1-C_b)\right]$$

and thus,

$$\frac{V_a}{V_{(Eyler)}} = \frac{1}{a\left[1 + V_R\,\frac{1-a}{a}\,(1-C_b)\right]} \qquad Eq\ 97$$

where, $V_R = \frac{V_b}{V_a}$.

Following Shook (1983, p. 33), we have the following equation, derived from the force balance for the lower layer:

$$j = \frac{2\,(s-1)\,C_b\,\mu_s\,(\sin\beta - \beta\,\cos\beta)}{(1-a)\,\pi} + \frac{f_0}{2\,\pi\,D\,g} \times \frac{V_b^2\,\beta - C_1\,\xi\,(V_a - V_b)^2\,\sin\beta}{(1-a)}$$

$$= \frac{j_p\,\phi_1}{(1-a)} + \frac{f_0\,V_a^2}{2\,D\,g} \times \frac{V_R^2\,\beta - C_1\,\xi\,(1-V_R)^2\sin\beta}{\pi\,(1-a)}$$

$$= \frac{j_p\,\phi_1}{(1-a)} + \frac{f_0\,V_a^2}{2\,D\,g} \times \frac{V_R^2\,\beta - \phi_2}{\pi\,(1-a)}.$$

On re-arranging, we get,

$$\frac{f_0\,V_a^2}{2\,D\,g} = \frac{\left[j_p\,\phi_1 - j\,(1-a)\right]\pi}{(\phi_2 - V_R^2\,\beta)}$$

or,

$$\frac{f_0\,V_a^2}{2\,D\,g\,j_p} = \frac{\left[\phi_1 - (1-a)\,Y\right]\pi}{(\phi_2 - V_R^2\,\beta)} \qquad \left(\text{where, } Y = \frac{j}{j_p}\right).$$

By slightly re-arranging Equation (A.14) of Eyler et al. (1982), we can write:

$$\frac{j_0}{j_p} \times \frac{\phi_3}{a^2} = \frac{\left[\phi_1 - (1-a)Y\right]\pi}{(\phi_2 - V_R^2\beta)} \qquad (\text{where } C_2 = 1).$$

By comparing the two previous equations,

$$\frac{j_0}{j_p} \times \frac{\phi_3}{a^2} = \frac{f_0\,V_a^2}{2\,D\,g\,j_p} = \frac{f_0\,V_{(Eyler)}^2}{2\,D\,g\,j_p} \times \frac{V_a^2}{V_{(Eyler)}^2}.$$

Now, if $j_0 = \frac{f_0\,V_{(Eyler)}^2}{2\,D\,g}$, then $\frac{\phi_3}{a^2} = \frac{V_a^2}{V_{(Eyler)}^2}$.

Therefore,

$$\phi_3 = a^2\,\frac{V_a^2}{V_{(Eyler)}^2} = \frac{1}{\left[1 + V_R\,\frac{1-a}{a}\,(1-C_b)\right]^2} \qquad (\text{from Equation 97}),$$

in accordance with Equation (A.18) of Eyler et al. (1982).

On the other hand, by Shook's interpretation, $V_{m(Eyler)}$ is to be substituted for $V_{(Eyler)}$. Then,

$$\frac{j_0'}{j_p} \times \frac{\phi_3'}{a^2} = \frac{f_0 V_a^2}{2 D g j_p} = \frac{f_0 V_{m(Eyler)}^2}{2 D g j_p} \times \frac{V_a^2}{V_{m(Eyler)}^2}.$$

Now, if $j_0' = \dfrac{f_0 V_{m(Eyler)}^2}{2 D g}$, then $\dfrac{\phi_3'}{a^2} = \dfrac{V_a^2}{V_{m(Eyler)}^2}$.

Hence,

$$\phi_3' = a^2 \frac{V_a^2}{V_{m(Eyler)}^2} = (1 - C_R)^2 \qquad \text{(from Equation 96).}$$

It should be noted that j_0 and j_0' have different values in the two different cases, when referring to the same set of data*.

In fact, $j_0 \times \phi_3 = j_0' \times \phi_3'$, and

$$\frac{j_0}{j_0'} = \frac{\phi_3'}{\phi_3} = \frac{V_{(Eyler)}^2}{V_{m(Eyler)}^2}$$

The relation between $V_{(Eyler)}$ and $V_{m(Eyler)}$ is simply

$$V_{(Eyler)} = V_{m(Eyler)} \times (1 - C_d)$$

since,

$$A \times V_{(Eyler)} = A \times V_{m(Eyler)} \times (1 - C_d)$$

gives the volumetric rate of delivery of the fluid.

It follows that the results of computations reported by Shook (1983, p. 19) cannot be compared directly with those reported by Eyler et al. (1982, p. A.19).

The concentration assumed is $C_d = 0.40$. Thus,

$$V_{(Eyler)} = V_{m(Eyler)} \times (1 - C_d) = 0.60\, V_{m(Eyler)}.$$

For example, $V_{(Eyler)} = 3.35\ m/s$, reported by Eyler et al. (1982), corresponds to $V_{m(Eyler)} = \dfrac{3.35}{0.60} = 5.58\ m/s$, whereas Shook (1983) compares this case with a case where $V_{m(Eyler)} = 3.35\ m/s$.

However, two approximate comparisons can be made, by matching the cases computed by Shook with cases selected from those reported by Eyler et al., as indicated in

* Because $V_{(Eyler)} \neq V_{m(Eyler)}$.

Tables 13 and 14. An appropriate comparison — as in Table (1) of Shook (1983, p. 19) — is given in Table 14.

TABLE 13

Selection of comparable cases on the basis of $V_{(Eyler)}$

Shook (1983)		Eyler et al. (1982)
$V_{(Shook)} = V_{m(Eyler)}$	$V_{(Eyler)} = (*)$	$V_{(Eyler)}$
3.35	2.01	2.45**
		1.76**
0.87	0.52	0.49

(*) $= (1 - C_d)\, V_{m(Eyler)}$.

** Use 2.10, i.e., the average of these two cases.

TABLE 14

Comparison of pipe-interface-angles, and of head losses, as computed by Shook and by Eyler, respectively

Slurry velocity (m/s)			Pipe-interface-angle (deg)		Head loss (m/m)	
$V_{(Shook)}$	$V^{\dagger}_{(Eyler)}$	$V^{\S}_{(Eyler)}$	$\beta^{\ddagger}$	β^{*}	$i_1^{\ddagger}$	i^{*}
		2.45		118		0.216
		1.76		120		0.171
3.35	2.01	2.10**	120	119**	0.182	0.198**
0.87	0.52	0.49	131 $\Longrightarrow$	130	0.143 $\Longrightarrow$	0.139

† See Table 13, Column 2.

§ See Table 13, Column 3.

‡ As computed by Shook.

* As computed by Eyler et al.

** Average of the two prior cases listed in this column.

$\Longrightarrow$ I.e., compare the two values indicated (e.g., 131 with 130).

Note: the agreement of the two sets of results shown above is considerably better than that in the comparison shown by Shook (1983, p.19, Table 1).

Discussion of the Use of the Durand Plot (With −1 Slope) as an Indication of the Applicability of the Wilson Model

As shown in Equation 4, when the slope of the Durand plot is −1, the following relation applies to the data:

$$\frac{i - i_w}{C_t\, i_w} = K\, \psi^{-1} \qquad Eq\ 98$$

or,

$$i - i_w = K\, C_t\, i_w\, \psi^{-1}. \qquad Eq\ 99$$

Substituting the usual expression for ψ (see Equation 3), we find that

$$i - i_w = K\, C_t\, i_w\, \frac{g\,(s-1)\,D}{V^2\,\sqrt{C_D}}. \qquad Eq\ 100$$

Since,

$$i_w = \frac{f_w\, V^2}{2\,g\,D}$$

Equation 100 becomes

$$i - i_w = \frac{K\, f_w}{2\,\sqrt{C_D}}\, C_t\,(s-1). \qquad Eq\ 101$$

Since K is a constant and C_D is also a constant (characteristic of any given solids in the slurry) and since f_w varies by a negligible amount over the usual range of V, we may write

$$i - i_w = K'\, C_t\,(s-1) \qquad Eq\ 102$$

where, K' is a constant.

Moreover, if the difference $(i - i_w)$ is attributed to the frictional force exerted by a sliding contact-load, we may also write

$$i - i_w = K_2\, C_d\,(s-1) \qquad Eq\ 103$$

as proposed by Newitt et al. (1955, Equation 18), and quoted by Shook et al. (1981, Equation 14), as well as by Clift et al. (1982, Equation 8).

Equations 102 and 103 are similar in form, but only when $m = 1$, and when any difference between C_d and C_t is ignored. It may be noted that Shook et al. (1981) ignore the effect of such a difference at this point, but on p. 90 they speculate on the difference between values of m as between use of C_d, or C_t, for the Durand plot.

The remark by Shook et al. (1981, p. 88) in relation to $m = 1$, that "this indicates the sliding-bed model may be applicable for coal of the size used here", seems to accept

agreement with the Newitt et al. (1955) equation as an indication of possible applicability of an improved theory by Wilson.

Wilson (1975, p. 34) refers to the Newitt et al. (1955) equation as an approximation to his own. His Figure (3) shows his curves for various values of $C_r = C_d/C_b$, and the corresponding straight lines of Newitt et al. (1955). At higher velocities, the curves and lines approach parallelism. The term *fully stratified* in the title of his Figure (3) indicates that no suspension is considered.

Curiously, the remark in question by Shook et al. (1981, p. 88) occurs in relation to data on coals having substantial proportions of fine and intermediate sizes, where suspension must occur. The remark is not strongly based.

Discussion of the Differentiation of the Quotient $\frac{j_h}{j_0 C}$, to Optimize the Value of Concentration C, for Efficient Pipelining Operation

The discussion relating to Equation (1) of the paper by Wilson and Judge (1980, p. 15) introduces an expression for the SEC (specific energy consumption), that states

$$SEC = \Omega \frac{j}{C S} \qquad Eq\ 104$$

where:

Ω is a constant,
j is the head loss in m/m, and
S is the specific gravity of the solids.

If a minimum of SEC exists, with respect to variation of C, then the value of C associated with it is found by solving the equation

$$\frac{d\,(j/C)}{d\,C} = 0. \qquad Eq\ 105$$

On p. 21 of the paper in question, the authors state that for efficient operation the optimum value of C should be used, and that this can be obtained by solving the equation

$$\frac{d}{d\,C}\left(\frac{j_h}{j_0\,C}\right) = 0 \qquad Eq\ 106$$

to optimize the *Specific Energy Consumption**.

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* Note that in this case:

j_h = hydraulic gradient due to the homogeneous component of the flow,
j_0 = hydraulic gradient due to the carrier fluid alone,
C = delivered concentration of the solids, by volume.

Equation 106 is not established rigorously, but the line of thought can be understood in light of the following points:

(1) on p. 19 (second last paragraph) three *types of optimum* are described, and it is stated that any type, other than type A, "will not be discussed further in the present article."

A clear statement is not made, but the authors appear to treat the type A case as if there were no significant contact-load when V is equal to, or greater than, V_L. It follows that $j = j_h$.

(2) The definition of V_L is given in Equation (4) of the paper by Wilson and Judge (1980), which states that

$$X_{sm} = \left(\frac{f_0}{2\,\mu_s\,C_b}\right)\left(\frac{V_L^2}{2\,g\,(s-1)\,D}\right) \qquad Eq\ 107$$

Note also the statement (fourth paragraph of p. 18): "The foregoing is intended simply to indicate the background of the analysis of contact-load transport." In fact, the only result of this analysis used in connection with type A cases is the value of V_L $\left(\text{or } F_L = \frac{V_L}{\sqrt{2\,g\,(s-1)\,D}}\right)$, given as a function of $\Delta = \frac{d}{C_D\,D}$ =*Suspension Parameter* in Figure (4) of that paper.

(3) Two paragraphs (last paragraph p. 19, and the following one p. 20), are confusing initially. First the authors speak of X_s as a function of X_t, i.e., in terms of contact-load theory. In these terms there is an *Optimum Operating Velocity* (say V_s, associated with X_s) for each value of C, i.e., V_s is a function of C.

But then they switch to V_L as the optimum velocity, independent of C. We can write $V_L = V_{sm}$, by analogy with V_s above.

In Figure 18, Y is plotted against V, for the curve defining the boundary of the *sationary deposit zone*, similarly to Figure (1) of Wilson and Judge (1980). For any value of C, say C_1, the *Optimum Operating Velocity* is considered to be V_s, i.e., Y (or j) has a minimum at V_s.

But, as an approximation V_L is used, instead of V_s, for the purpose of finding the value of C that gives the efficiency criterion SEC a minimum value.

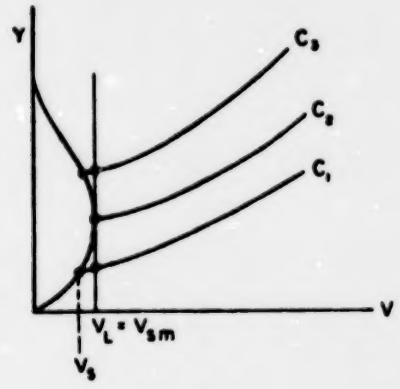

Fig. 18 — Y against V plot for curve defining boundary of stationary deposit zone

In view of the foregoing points, and also noting that C_{rh}, the *Relative Concentration of Solids in Suspension*, is

$$C_{rh} = C_r - C_{rc} = \frac{C}{C_b} - \frac{C_c}{C_b} = \frac{C - C_c}{C_b} = \frac{C_h}{C_b}$$

we can write Equation (13) of Wilson and Judge (1980) as

$$\mu_r = \frac{exp\,(-1.44\,C_{rh})}{(1 - C_{rh})^3} = \frac{exp\,(-1.44\,C_h/C_b)}{[1 - (C_h/C_b)]^3} = \frac{exp\,(-K_1\,C)}{[1 - (C/C_b)]^3} \qquad Eq\ 108$$

where:

C_b is the volumetric fraction of solids in a loose-packed bed (a constant),

C is the delivered concentration of solids, by volume, and

$K_1 = 1.44/C_b$.

We can also write Equation (11) of that paper as

$$\frac{j_h}{j_0} = \mu_r^m\ [1 + (s - 1)\,C_b\,C_{rh}]^{(1-m)}$$

$$= \mu_r^m\ [1 + (s - 1)\,C_h]^{(1-m)} = \mu_r^m\ [1 + (s - 1)\,C]^{(1-m)} \qquad Eq\ 109$$

where, j_0, m and s are constants. It will be noted (from Equation 107) that j_0, i.e., the head loss for the carrier fluid alone, is associated with V_L.

It follows that $\frac{j_h}{j_0 C}$ is a function of C only, and that it can be differentiated as follows:

$$\frac{d\,\mu_r}{d\,C} = \frac{d}{d\,C}\,\frac{exp\,(-K_1\,C)}{[1-(C/C_b)]^3}$$

$$= exp\,(-K_1\,C)\,\frac{d}{d\,C}\left[\frac{1}{[1-(C/C_b)]^3}\right] + \frac{1}{[1-(C/C_b)]^3}\,\frac{d}{d\,C}\,exp\,(-K_1\,C)$$

$$= \frac{-3\,exp\,(-K_1\,C)}{[1-(C/C_b)]^4}\left(\frac{-1}{C_b}\right) - \frac{K_1\,exp\,(-K_1\,C)}{[1-(C/C_b)]^3}$$

$$= \frac{3\,exp\,(-K_1\,C)}{C_b\,[1-(C/C_b)]^4} - \frac{K_1\,exp\,(-K_1\,C)}{[1-(C/C_b)]^3}$$

and,

$$\frac{d}{d\,C}\left(\frac{j_h}{j_0}\right) = \mu_r^m\,\frac{d}{d\,C}\,[1+(s-1)\,C]^{(1-m)} + [1+(s-1)\,C]^{(1-m)}\,\frac{d\,\mu_r^m}{d\,C}$$

$$= \mu_r^m\,(1-m)\,(s-1)\,[1+(s-1)\,C]^{-m} + m\,[1+(s-1)\,C]^{(1-m)}\,\mu_r^{(m-1)}\,\frac{d\,\mu_r}{d\,C}.$$

Then, $\frac{d}{d\,C}\left(\frac{j_h}{j_0 C}\right)$ can be expressed in terms of the above expressions, as follows:

$$\frac{d}{d\,C}\left(\frac{j_h}{j_0\,C}\right) = \frac{j_h}{j_0}\,\frac{d}{d\,C}\left(\frac{1}{C}\right) + \frac{1}{C}\,\frac{d}{d\,C}\left(\frac{j_h}{j_0}\right)$$

$$= \frac{1}{C}\,\frac{d}{d\,C}\left(\frac{j_h}{j_0}\right) - \frac{1}{C^2}\left(\frac{j_h}{j_0}\right)$$

Since j_0 is a constant, the solution of Equation 106 is equivalent to the solution of Equation 105.

Note that the value of C found by solving Equation 106 is quite independent of the value C_2 (shown on Figure 18) that is associated with V_{sm} (or X_{sm}).

Discussion of the Ergun Equation for Flow Through Porous Media

Shook (1983, p. 10) states that the Ergun equation for flow through porous media can be generalized as

$$(f_s)_f = C_{fb}\,\rho_f\,\frac{(V_f - V_s)\,|\,V_f - V_s\,|}{d} \qquad \text{(Shook, 1983, Eq 7)}$$

where,

C_{fb} = friction factor for fluid flowing through contact load particles (i.e., through the bed)

$$= \left[\frac{150\mu_f C}{d\rho_f \mid V_f - V_s \mid (1-C)}\right] + 1.75 \qquad \text{(Shook, 1983, Eq 8)}$$

Similarly, Eyler et al. (1982, p. 4.12) give the form of the Ergun equation as follows:

$$j = \left[\frac{150C_b\mu}{D_p} + 1.75G\right]\left[\frac{C_b}{(1-C_b)^3}\right]\left[\frac{G}{D_p\rho^2 g}\right] \qquad \text{(Eyler et al., 1982, Eq 4.19)}$$

where:

$$G = \text{superficial mass velocity} \quad \left(\frac{\text{kg}}{\text{s} \times \text{m}^2}\right)$$

$$D_p = \text{equivalent particle diameter} = 6 \times \frac{\text{particle volume}}{\text{particle surface area}}.$$

Also,

$$D_p \equiv d_{(Shook)}$$

$$C_b \equiv C_{(Shook)} = \text{volumetric concentration of loose} - \text{packed solids,}$$

and,

$$\frac{dP}{dz} = j \times \rho \times g.$$

The question, therefore, arises as to the equality of the two different forms of the Ergun equations quoted, namely Shook's (1983) Equation (7), and Eyler's (1982) Equation (4.19).

For Equations (6) and (7) of Shook (1983, p. 10), the assumptions are stated: (a) that

$$C\,(f_s)_f + (1-C)\,(f_f)_s = 0 \qquad \text{(Shook, 1983, Eq 6)}$$

and, hence, that $(f_f)_s = -\frac{C}{(1-C)}(f_s)_f$;

also (b), since $V_s = 0$, that

$$(f_s)_f = C_{fb}\,\rho_f\,\frac{v_f^2}{d}. \qquad \text{(Shook, 1983, Eq 7)}$$

Moreover, (c) that

$$f_{fb} = 0 \quad \text{and} \quad \frac{\partial V_1}{\partial t} = 0 = \frac{\partial V_1}{\partial z}.$$

It should also be noted, that Shook (1983, p. 10) refers to v_f as the *true fluid velocity*, while calling $v_f(1-C)$ the *superficial fluid velocity*, in (m/s). Eyler, on the other hand, refers to G as the *superficial mass flowrate*, with dimensions

$$\left(\text{kg} \times \frac{\text{m}}{\text{s}} \times \frac{1}{\text{m}^3}\right) = \left(\frac{\text{kg}}{\text{s} \times \text{m}^2}\right).$$

The dimensions of $\frac{G}{v_f(1-C)}$ are thus, $\frac{kg}{s \times m^2} \times \frac{1}{m/s} = \frac{kg}{m^3}$; this expression, represents the dimension of a *density*, in this case ρ_f. Consequently, $G = \rho_f\, v_f\, (1-C)$. From Shook (1983, p. 9, Equation 1) it follows that

$$\frac{dP}{dz} = f_1 = f_f = (f_f)_s$$

because, from Shook's Equation (3), $b_1 = 0$ (since $\frac{\partial h_1}{\partial z} = 0$ for a horizontal pipeline); also, from Shook's Equation (4), $f_f = (f_f)_s$, since f_{fb} =boundary force on the fluid = 0, (p. 10 of Shook), and also since, for steady flow (again, Shook, 1983, p. 10),

$$\frac{\partial V_1}{\partial t} = \frac{\partial v_1}{\partial z} = 0.$$

Now, from Eyler et al. (1982, p. 4.12), namely from their Ergun equation (Equation 4.19), with $G = \rho_f\, v_f\, (1-C)$, we get the following expression for $(f_f)_s$:

$$\begin{aligned}
(f_f)_s = \frac{dP}{dz} &= j \times \rho_f \times g \\
&= \rho_f\, g \left\{ \left[\frac{150 C \mu}{d} + 1.75\, \rho_f\, v_f\, (1-C) \right] \times \frac{C}{(1-C)^3} \times \left[\frac{\rho_f v_f (1-C)}{d \rho_f^2 g} \right] \right\} \\
&= \left[\frac{150\, C\, \mu}{d\,(1-C)} + 1.75\, \rho_f\, v_f \right] \times \frac{C}{(1-C)} \times \left[\frac{v_f}{d} \right] \\
&= \rho_f\, v_f \left[\frac{150\, C\, \mu}{\rho_f v_f d (1-C)} + 1.75 \right] \times \frac{C}{(1-C)} \times \frac{v_f}{d} \\
&= C_{fb} \times \frac{C}{(1-C)} \times \frac{\rho_f\, v_f^2}{d}.
\end{aligned}$$

Hence,

$$(f_s)_f = -\frac{(1-C)}{C}\,(f_f)_s = -\frac{(1-C)}{C} \times C_{fb} \times \frac{C}{(1-C)} \times \frac{\rho_f v_f^2}{d} = -C_{fb}\, \frac{\rho_f\, v_f^2}{d}$$

Consequently, this proves the identity of Shook's and Eyler's expressions for the Ergun equation, because, aside from the negative sign, the foregoing equation agrees with Shook's (1983) Equation (7), for the case of $V_s = 0$ (i.e., for the case of a stationary bed).

SUMMARY

A study of some aspects of the Wilson model for slurry flow, including its connection with the formulae of Durand and of Newitt et al., concludes that the model should be treated as a new formulation for the empirical correlation of data. Difficulties involved in considering the parameters of the model, such as the mechanical friction factor η and the interfacial friction factor C_{f12}, as physical properties are discussed. It is shown that values for the threshold velocity V_t can be derived, as a function of the concentration of solids, from experimental data, with a possible improvement to the model.

Some comparative calculations relating to the Wilson model illustrate the equivalence of Wilson's and of Shook's use of the force balance equations, to a good approximation, under the same basic assumptions. Recognition of a difference between the definitions of Wilson's ζ and Eyler's ξ emerged from a connected study. It is also shown that a correct solution of the equations of the model is highly sensitive to the value of the interfacial friction factor, although the value of the hydraulic gradient is not. A simple means for fitting ζ to a set of data, when all other parameters are known or assumed, is given.

A plan for data accumulation, based on dimensional analysis, is proposed. An introduction to dimensional analysis for slurry flow in pipes is appended. The value of an integrated approach to the construction of dimensionless variables is discussed.

The results of short studies of some theoretical and practical topics encountered in the literature relating to the Wilson model are presented. The study of the *Buoyancy Effect on Contact Particles* concludes that the Wilson model is open to improvement by research to determine the friction factor, η, not as a constant, but as a function, dependent on the concentration of solids and on other variables.

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APPENDIX A

Some Basic Features of Fluid Flow

Some Basic Features of Fluid Flow

The theory of engineering fluid mechanics can be found in numerous standard text books, e.g., Jaeger (1956), Daugherty and Franzini (1977). An introduction to the results of more recent work on the nature of fluid flow is given by Tennekes and Lumley (1972). These authors attempt to bridge the gap between standard texts, and advanced texts on turbulence.

At a sufficiently low rate of flow in a straight unobstructed pipe, the particles of fluid move in laminar flow, i.e., in straight lines parallel to the axis. The velocity is a maximum at the axis, and is zero at the wall. Thus the body of the fluid is subjected to shear at various rates, depending on the distance from the axis of the pipe.

The capacity of the fluid to dissipate energy under shear is represented by the coefficient of viscosity, μ. The pressure gradient, expressed as the gradient of head of fluid in *metres per metre* length of pipe is given by

$$i = f \frac{V^2}{2 g D} \qquad Eq\ (A-1)$$

where:

V is the mean velocity of the fluid,
D is the diameter of the pipe,
g is the acceleration due to gravity,

and

$$f = 64 \frac{\mu/\rho}{D V} = \frac{64}{R_e} \qquad Eq\ (A-2)$$

where:

ρ is the density of the fluid,
$R_e = \frac{D V \rho}{\mu} = \frac{D V}{\nu}$ is the Reynolds number of the flow,
$\nu = \mu / \rho$ is the kinematic viscosity, and
f is the Darcy-Weisbach hydraulic friction factor.

f must be regarded as a mean internal friction factor, because energy is dissipated throughout the body of flowing fluid as a function of the local rates of shear.

When the rate of flow is increased (with increasing Reynolds number), the laminar condition is replaced by an unstable condition, and eddies are generated by the main shear

flow. For unstable, or turbulent, flow in a pipe of circular section, the friction factor for use in Equation (A-1) is given implicitly by the Colebrook-White formula

$$\frac{1}{\sqrt{f}} = -2\,log\left(\frac{k}{3.7\,D} + \frac{2.51}{Re\sqrt{f}}\right) \qquad Eq\ (A-3)$$

where:

k is the wall roughness size, and *log* indicates logarithm to base 10.

When R_e is large enough to make the second term in brackets negligible, the energy loss depends only on the roughness ratio, k/D, and is independent of μ. Such a flow is said to be *rough turbulent.*

But k and R_e may have such values that the first term is negligible. Then, only the second term remains, and the flow is said to be *smooth turbulent.* However, R_e must be greater than about 1000, the upper limit of the range in which laminar flow is stable. Otherwise laminar flow will prevail, and Equation (A-2) will apply.

No matter how small k/D may be (but not zero), R_e may be made large enough for the flow to become *rough turbulent*, as indicated above. With increasing values of R_e, f will approach a constant value characteristic of the value of k/D, and the flow may be termed to be *fully turbulent.*

A rhyme by L.F. Richardson, dating from the 1920's, showed a remarkable intuitive insight into the nature of turbulent flows. It is quoted by Yalin (1977, p. 90) as follows:

"Big whorls have little whorls,
Which feed on their velocity;
Little whorls have smaller whorls,
And so on unto viscosity."

Only years later was substantial progress made in mathematical treatments of the processes expressed in this rhyme.

Tennekes and Lumley (1972) describe the principal methods that have been used to study turbulence. One of the most powerful methods is dimensional analysis, an application of which is described in Appendix B. The method is useful for planning experimental studies, but it has also been used as the basis of complex theoretical concepts.

The elements of some (if not all) theories are the sizes of the eddies in the flow, and their rates of rotation, or their frequencies. According to ideas used by Prandtl and von Karman, the size of an eddy is proportional to its distance from the flow boundary. But a statistical approach views the size of such an eddy as the mean of a random distribution

of sizes. A related approach studies the distribution of frequencies of rotation of such an eddy, including its standard deviation.

In spite of the efforts that have been made to understand turbulence in detail, no adequate model of turbulence yet exists. Partial improved understanding has been achieved, but different approaches do not fully agree.

However, an important feature of turbulent flow, recognized in the foregoing rhyme, can be illustrated by a result from the statistical approach. Accordingly, a description given by Tennekes and Lumley (1972) will be sketched here briefly.

In an unstable condition ($R_e > 2000$), the main shear flow transfers energy to eddies with diameters of the order of the radius of the pipe. Some energy is dissipated by shearing in these eddies. A cascade process is set up, in which energy is passed on to smaller and smaller eddies, until it is all dissipated by the cumulative viscous loss. At high rates of flow, when the turbulence is vigorous, the larger eddies transfer most of their energy to slightly smaller eddies, dissipating only a small part in direct shear. Most of the energy is dissipated in the smallest eddies (cascade process). When the rate of flow is increased, more energy goes into the largest eddies, and the eddies in which most of the energy is finally dissipated are even smaller than formerly. Under these conditions the rate of dissipation of energy is determined by the inertial coupling between the main shear flow, and the large eddies. The rate of energy loss becomes independent of viscosity, even though the energy is ultimately dissipated in viscous shear in the smallest eddies.

Consequently, the state of turbulence is determined by the shape of the boundary walls, including their roughness, and not by the fluid properties directly. Turbulence is said not to be a feature of fluids, but of fluid flows. Turbulent flows in similar channels or conduits are similar, irrespective of the fluid, whether liquid or gas, if the Reynolds number is large enough. All turbulent flows have many characteristics in common, but they may differ in important ways when their boundaries are dissimilar.

However, the cascade process is only the mean result of the actual physical process, which is not considered in detail. A brief summary by Yalin (1977, p. 204) describes some results of recent experimental work that provide some knowledge of the physical details of turbulent flow. This approach recognizes that some sort of short-term order must exist in the flow, rather than complete randomness.

A mathematical area known as *catastrophe theory* (Kadanoff, 1983) is concerned with systems of equations that exhibit aspects both of regularity and of randomness. Simple equations that govern some processes deterministically under certain conditions can, under slightly different conditions, lead to a chaotic process. An underlying pattern may remain, however. Such systems offer a possibility of applications in studies of fluid flow.

This sample of formulae and theories as discussed, like others, deals only with fluids. The addition of solid particles to the flow compounds the complexities that must be dealt with.

In case of large-particle transport, with the particles concentrated toward the invert of the pipe, and moving more slowly than the fluid, the solids become part of the boundary of the fluid component, albeit a moving boundary. The roughness of the boundary is then well beyond the range of roughnesses that has been studied extensively. The shape of the channel for fluid flow is far from that of a simple circle.

Consequently, the adaptation of standard formulae for use with such flows presents a difficult challenge. Any proposals must be tested against experiment with great care, under all ranges of parameters that are of interest. Much of the present report was written with such thoughts in mind.

APPENDIX B

Introduction to Dimensional Analysis for Flow in Pipes

Introduction to Dimensional Analysis for Flow in Pipes

Dimensionless expressions are frequently encountered in reports on theoretical and experimental studies of problems in hydraulics, but often they have not been used to best effect through the lack of an integrated approach to their construction. Consequently, it was of real interest to consider the basics of this concept within the framework of this cooperative research project.

Yalin (1977) gives an excellent account of the employment of dimensionless expressions, principally in connection with the transport of sedimentary materials in open channels. What follows consists of an application of Yalin's system to slurry flow in pipes, as interpreted by this report's second author. Yalin's notation is followed as closely as possible, as it is assumed that readers will wish to make further references to his book, apart from the information imparted by noting the following section of this report. However, some changes are needed for the sake of compatibility with notations commonly used in slurry transport.

A minimum list of parameters characteristic of slurry flow in a straight horizontal pipe is given in Table (B-1). The fluid is assumed to have the simple Newtonian properties specified by ρ and μ. Particle properties are specified by ρ_s and d. For simplicity, we assume, for the present, that the particle size is closely graded, so that the length d is sufficiently representative of the mechanical behavior of the particles. Additionally, the volume concentration of particles is designated by C.

The pipe properties are specified by D and k. The direct frictional interaction between pipe and particles is specified by η.

The driving pressure of the flow is specified by i, the fluid head gradient in metre per metre. Alternatively, the rate of flow could be specified by V, the mean velocity, but it will appear that use of i is particularly convenient.

The gravitational constant, g, does not affect the flow of clear fluids under pressure in pipes, in general, but because gravity has a differential action between particles and fluid of different densities, g is a characteristic parameter of slurry flow.

All properties of the flow are determined by a complete list of characteristic parameters. For example, the rate of flow, V, can be expressed as a function of the parameters, in principle, i.e.,

$$V = f_V(\rho, \mu, \rho_s, d, C, D, k, \eta, i, g). \qquad Eq\ (B-1)$$

At this point it is advantageous to consider ways of combining some of the parameters into groups that can be used as new parameters, to replace some of the original parameters.

TABLE (B-1)

Minimum list of characteristic parameters for slurry flow in pipe

Characteristic Parameters	Symbol	Units	Dimensions
Fluid density	ρ	kg/m^3	ML^{-3}
Fluid viscosity	μ	kg/ms	$ML^{-1}T^{-1}$
Particle density	ρ_s	kg/m^3	ML^{-3}
Particle diameter	d	m	L
Concentration of particles	C	—	1
Pipe diameter	D	m	L
Pipe roughness	k	m	L
Friction coefficient, particle - pipe	η	—	1
Fluid head gradient	i	—	1
Gravitational constant	g	m/s^2	LT^{-2}

The shear or friction velocity is given by

$$v_* = \sqrt{\frac{igD}{4}} = V\sqrt{\frac{f}{8}} = \sqrt{\frac{dP}{dz} \times \frac{D}{4\rho}} \qquad Eq\ (B-2)$$

because $\Delta P/L = dP/dz = ig\rho$. Also $i = fV^2/2gD$ (with f = the Darcy-Weisbach friction factor).

The pressure gradient, dP/dz, is normally created by pumping, and has nothing to do with gravity. Thus v_* is not a function of gravity. Gravity appears in the expression $v_* = \sqrt{igD/4}$ only because it appears in the relation between i and dP/dz. This anomaly is due to the use of i for convenience, although it is not expressed in fundamental units. All other characteristic parameters are expressed in fundamental units.

This point is emphasized, because another new parameter, γ_s, is used to represent the submerged weight of particles, i.e., $\gamma_s = (\rho_s - \rho)g$. The action of gravity on the flow is completely represented by γ_s. No other combination of parameters with g is required. Thus, it is convenient to replace g by γ_s in the parameter list. ρ_s and ρ remain in the list, because γ_s can replace only one of the parameters of which it is composed.

Any such substitution must satisfy the conditions that the number of parameters remains the same, and that all parameters in the new list be independent.

Similarly, v_* is used to replace i, which is convenient, because v_* is involved in fundamental relations of hydraulic theory. We may, therefore, re-write Equation (B-1) as

$$V = f_V(\rho, \mu, \rho_*, d, C, D, k, \eta, v_*, \gamma_*). \qquad Eq\ (B-1a)$$

But, it is shown by dimensional analysis that a function such as f_V can be determined by a number of dimensionless variables, three less than the number of parameters. Any three characteristic parameters that are dimensionally independent can be chosen as basic parameters. We may choose, for example, d, ρ, v_*. They are independent dimensionally, because no product of the form $(d^a \rho^b v_*^c)$ can be formed that is dimensionless.

We can form a dimensionless product with another parameter, as indicated in the following example. Let

$$[d^a\ \rho^b\ v_*^c\ \mu^{-1}] = M^0 L^0 T^0 \equiv\ 1.$$

This equation reads: the dimensions of $[d^a\ \rho^b\ v_*^c\ \mu^{-1}]$ are zero in mass (M), zero in length (L), and zero in time (T). Then,

$$L^a . \left(\frac{M}{L^3}\right)^b \left(\frac{L}{T}\right)^c \left(\frac{M}{LT}\right)^{-1} = M^0 L^0 T^0$$

The equation given by the exponents of L is:

$$(a - 3b + c + 1) = 0.$$

Similarly, for M, it is $(b-1) = 0$; and similarly for T, it is $(-c+1) = 0$.

Hence, $b = 1$; $c = 1$; and, $a = (3b - c - 1) = 1$.

Therefore, the dimensionless product representing μ is

$$\frac{d\rho v_*}{\mu} = \frac{v_* d}{\nu}.$$

Note that the choice of either $+1$ or -1 for μ was quite arbitrary.

We use the basic parameters to form a dimensionless product (called a *variable*) with each of the other parameters. Table (B-2) gives the revised list of parameters, with their dimensions, together with the dimensionless variables.

It will be noticed that the dimensionless variables X_5 and X_7, representing C and η respectively, are simply C and η, because the latter are dimensionless in themselves.

TABLE (B-2)

Revised list of characteristic parameters for slurry flow in pipe, with dimensionless variables and with basic parameters d, ρ, v_*

Characteristic Parameters		Variable	
Symbol	Dimensions	Product	Symbol
ρ	ML^{-3}	*basic*	—
μ	$ML^{-1}T^{-1}$	$\frac{v_* d}{\nu} = \frac{v_* d}{\mu/\rho}$	$X(X_1)$
ρ_*	ML^{-3}	ρ_*/ρ	$W(X_4)$
d	L	*basic*	—
C	1	C	X_5
D	L	D/d	$Z(X_3)$
k	L	k/d	X_6
η	1	η	X_7
v_*	LT^{-1}	*basic*	—
γ_*	$ML^{-2}T^{-2}$	$\frac{\rho v_*^2}{\gamma_* d}$	$Y(X_2)$

The variable $X = \frac{dv_*}{\nu}$, representing μ, is a Reynolds number, known as the grain-size or particle Reynolds number. It is appropriate for use in particle transport studies.

The variable Y is a type of Froude number, because it can be written as

$$Y = \frac{\rho v_*^2}{\gamma_* d} = \frac{\rho v_*^2}{g(\rho_* - \rho)d} = \frac{v_*^2}{gd(s-1)} = \frac{(F_r')^2}{(s-1)}$$

where,

$$F_r' = \frac{v_*}{\sqrt{gd}} = \frac{V}{\sqrt{gD}} \times \frac{v_*}{V} \times \sqrt{\frac{D}{d}} = F_r \times \frac{v_*}{V} \times \sqrt{\frac{D}{d}}.$$

The variable Z is the ratio of pipe diameter to typical particle diameter; W is the ratio of particle density to fluid density; and X_6 is the ratio of pipe roughness to particle diameter. The pipe roughness ratio is given by $k/D = X_6/Z$.

As another example, we may again choose d, ρ, v_* as the basic parameters, but γ as another parameter. In this case:

$$[d^a \rho^b v_*^c \gamma_*^{-1}] = M^0 L^0 T^0.$$

Then,

$$\left[L^a\left(\frac{M}{L^3}\right)^b\left(\frac{L}{T}\right)^c\left(\frac{M}{L^2T^2}\right)^{-1}\right] = M^0L^0T^0.$$

Hence, the equation given by the exponents of L is:

$$(a - 3b + c + 2) = 0$$

Similarly, for M, it is $(b - 1) = 0$; and again for T, it is $(-c + 2) = 0$.

Hence, $b = 1$; $c = 2$; and $(a - 3 + 2 + 2) = 0$, i.e., $a = -1$.

Therefore, the dimensionless product representing γ_* is

$$[d^{-1}\rho v_*^2\gamma_*^{-1}] = \frac{\rho v_*^2}{\gamma_* d} = Y.$$

Now, any property A of the flow can, in the same manner as a characteristic parameter, be represented by a dimensionless expression of the form

$$\Pi_A = d^a\rho^b v_*^c A. \qquad Eq\ (B-3)$$

Furthermore, it can be shown that any Π_A can be expressed as a function of the dimensionless variables of the flow, i.e.,

$$\Pi_A = \phi_A(X, Y, Z, W, X_5, X_6, X_7). \qquad Eq\ (B-4)$$

In the case of the mean velocity, V, of the flow, $\Pi_V = V/v_*$ and we can write

$$\Pi_V = \phi_V(X, Y, Z, W, X_5, X_6, X_7). \qquad Eq\ (B-5)$$

This type of expression has the following advantages:

- Equation (B-1), involving ten parameters, has been replaced by Equation (B-5), involving only seven variables.
- Since Π_V and the seven variables are dimensionless, their numerical values do not depend on the system of units used. The relation in question is expressed in a universal form.
- The variables, X_i, are the *criteria of similarity* between systems operating at different size scales.

Let us now consider the problem of predicting the performance of a proposed slurry pipeline (prototype), on the basis of data obtained relative to a smaller-diameter slurry pipeline (model). Reference may be made to Yalin (1971), whose system of notation we also follow in this area.

Let us distinguish the parameters of the prototype pipeline flow by a *prime*, e.g., D' for diameter, and those of the model pipeline flow by double *primes*, e.g., D''. Let the scale factors relating the values of model parameters to those of prototype values be designated by λ_D, etc., i.e.,:

$$\lambda_D = D''/D'$$

$$\lambda_\rho = \rho''/\rho' \quad \text{etc.}$$

Complete dynamic similarity between the prototype and the model pipeline requires geometric similarity, i.e., all linear dimensions, including particle diameter and pipe roughness, must be reduced by the same factor. Then, the necessary, and sufficient, conditions for full dynamic similarity are that the corresponding dimensionless variables of the prototype, and of the model pipeline, be equal, i.e., that:

$$X_i'' = X_i', \quad with \quad i = 1, 2, 3, ..., 7. \qquad Eq\ (B-6)$$

Considering each variable in turn, we obtain:

(a) $\dfrac{v_*'' d''}{\nu''} = \dfrac{v_*' d'}{\nu'}$ relative to X (or X_1)

or $\dfrac{v_*''}{v_*'} \times \dfrac{d''}{d'} = \dfrac{\nu''}{\nu'}$ i.e., $\lambda_{v_*} . \lambda_d = \lambda_\nu = \dfrac{\lambda_\mu}{\lambda_\rho}$ Eq (B-7)

(b) $\dfrac{\rho''(v_*'')^2}{\gamma_*'' d''} = \dfrac{\rho'(v_*')^2}{\gamma_*' d'}$ relative to Y (or X_2)

i.e., $\lambda_\rho . \lambda_{v_*}^2 = \lambda_{\gamma_*} . \lambda_d$ Eq (B-8)

(c) $\dfrac{D''}{d''} = \dfrac{D'}{d'}$ relative to Z (or X_3)

i.e., $\lambda_D = \lambda_d$ Eq (B-9)

(d) $\dfrac{\rho_f''}{\rho''} = \dfrac{\rho_f'}{\rho'}$ relative to W (or X_4)

i.e., $\lambda_{\rho_*} = \lambda_\rho$ Eq (B-10)

(e)	$C'' = C'$		relative to X_5	
		i.e.,	$\lambda_C = 1$	Eq (B-11)
(f)	$\frac{k''}{d''} = \frac{k'}{d'}$		relative to X_6	
		i.e.,	$\lambda_k = \lambda_d$	Eq (B-12)
(g)	$\eta'' = \eta'$		relative to X_7	
		i.e.,	$\lambda_\eta = 1$	Eq (B-13)

Equations (B-9) and (B-12) are the basic conditions for geometric similarity between the prototype and model flows, i.e.,

$$\lambda_k = \lambda_d = \lambda_D$$

Equation (B-11) can be considered as an auxiliary condition for geometric similarity, because the particle distribution patterns in the prototype and model can only be similar if their total concentrations are the same. But all conditions for dynamic similarity must also be fulfilled if this aspect of geometric similarity is to be assured.

Equation (B-13) requires the coefficient of friction between the particles and the pipe wall to be the same in the prototype and model.

Normally water is the fluid used in both prototype and model. Thus fluid density and viscosity are identical in both, i.e., $\lambda_\rho = 1$, and $\lambda_\mu = 1$. It follows from Equation (B-10) that $\lambda_{\rho_s} = \lambda_\rho = 1$. The particle densities in prototype and model must be identical.

It also follows from Equation (B-7) that

$$\lambda_{v_s} \times \lambda_d = 1. \qquad Eq\ (B-14)$$

Since gravity is the same for both prototype and model, $\lambda_g = 1$. Since $\lambda_{\rho_s} = \lambda_\rho = 1$, and since $\gamma_s = g(\rho_s - \rho)$,

$$\lambda_{\gamma_s} = \lambda_g \times \lambda_{(\rho_s - \rho)} = 1 = \lambda_g \times \lambda_\rho = 1.$$

Thus it follows from Equation (B-8) that

$$\lambda_{v_s^2} = \lambda_d. \qquad Eq\ (B-15)$$

Now Equations (B-14) and (B-15) cannot be satisfied simultaneously, unless $\lambda_{v_*} = \lambda_d = 1$. Therefore, $\lambda_D = 1$, which means that similarity of flows in prototype and model cannot exist, in general, under the assumptions made, unless the pipe sizes are identical.

This situation is a common one relative to problems in hydraulics, but sometimes reduced-scale models can be operated usefully within a limited range of flow rates. Complete similarity may not exist, but if the value of a dependent variable, such as $V/v_* = \Pi_V$, is not a function of a given variable X within a certain range of X values, the equation $X' = X''$ (see Eq B-6) may be ignored when X' and X'' are within that range, i.e., the criterion of similarity, X, is not effective within that range.

Furthermore the criteria of similarity, or dimensionless variables, are useful in planning to obtain data and to derive empirical relationships between flow systems at different size scales, even if a variable like X is only partially effective (i.e., only in a high R_e range, at nearly turbulent flow). To explain their use, however, it seems to be convenient to assume, initially, that similarity can exist between prototype and model. Although the terms *prototype* and *model* will be retained in this report, both can refer to laboratory pipelines, the former with larger diameter, and the latter with smaller.

Because the transport of large particles in large pipelines requires a strongly turbulent flow, and because friction between particles and pipe wall is an important cause of energy loss, it seems reasonable to commence with the assumption that turbulence and mechanical friction (as opposed to hydraulic friction) will account for all of the energy loss in the flow. Also, it will be assumed that the ratio between the diameters of prototype and model will not be large. Because large-sized laboratory pipelines are available, it appears that a limited scale-up factor may serve a useful purpose for studying realistic field prototypes, which should ease the requirement for similarity between pipe roughness ratios of prototype and model.

The following initial hypotheses summarize the above assumptions:

- The variable X, i.e., X_1, representing the direct contribution of μ to energy loss, will be dropped from Equation (B-5), since Π_V does not depend on μ in case of fully turbulent flow.
- The variable $X_6 = \frac{k}{d}$ will be dropped from Equation (B-5).
- The size scale factor $\lambda_D = D''/D'$ will not be less than the order of 0.7. Smaller factors would make the first two hypotheses less acceptable.

As a result of these hypotheses, Equation (B-5) may be written as follows:

$$\Pi_V = \frac{V}{v_*} = \phi_V(Y, Z, W, X_5, X_7) = \phi_V\left(\frac{\rho v_*^2}{\gamma_s d}, \frac{D}{d}, \frac{\rho_s}{\rho}, C, \eta\right). \qquad Eq\ (B-16)$$

To produce a flow in the model closely similar to the flow in the prototype, we must satisfy five criteria of similarity, as deduced above, namely that:

$\lambda_\eta = 1$	see Eq (B-13)
$\lambda_C = 1$	see Eq (B-11)
$\lambda_{\rho_s} = \lambda_\rho = 1^*$	see Eq (B-10)
$\lambda_D = \lambda_d$	see Eq (B-9)
$\lambda_{v_*^2} = \lambda_d$	see Eq (B-15)

* $\lambda_\rho = 1$ since the same fluid (water) is assumed to be used in both prototype and model.

The simplest way to satisfy $\lambda_{\rho_s} = 1$ is to use the same particulate material in the model as that intended for the prototype. By using the same particulate material and the same pipe material, we may also satisfy $\lambda_\eta = 1$. The condition $\lambda_C = 1$ is simply satisfied by employing the same particulate concentration in prototype and model.

On the assumption that all series of runs will be carried out using similar materials, we may reduce Equation (B-16) to

$$\Pi_V = \frac{V}{v_*} = \phi_V\left(\frac{\rho v_*^2}{\gamma_s d}, \frac{D}{d}, C\right) \qquad Eq\ (B-17)$$

or,

$$\Pi_V = c = \phi_V(Y, Z, C)^{\ddagger}. \qquad Eq\ (B-18)$$

As mentioned earlier, the particles were assumed to be closely graded in size. If this is not the case, let us assume that the size distribution (Yalin, 1977, p. 7) must be specified by a number of sizes, say five, d_i, with $i = 1, 2, ..., 5$. Then, the condition $\lambda_D = \lambda_d$ becomes

$$\lambda_D = \lambda_{d_i}, \quad i = 1, 2, ..., 5$$

---

‡ Since $\frac{V}{v_*} = \sqrt{\frac{2}{C_f}} = \sqrt{\frac{8}{f}} = c$ where: C_f = Fanning friction factor, and f = Darcy-Weisbach friction factor; it is seen that c, used by Yalin, is also a friction factor.

or,

$$\frac{D''}{D'} = \frac{d''_1}{d'_1} = \frac{d''_2}{d'_2} = \frac{d''_3}{d'_3} = \frac{d''_4}{d'_4} = \frac{d''_5}{d'_5}. \qquad Eq\ (B-19)$$

In other words, the model sample of particles must have a size distribution similar to that of the prototype sample. In fact, any geometric particle characteristics, especially shapes and their distributions, must be similar for prototype and model. More generally, this condition must be assumed to be fulfilled in cases of all unspecified parameters (Yalin, 1977, Sec.1.3 and 3.2).

It is now necessary to consider a method for testing Equation (B-18). Let us assume that *prototype* and *model* pipelines of diameters D' and D'', respectively, are available. We also assume that a particulate material, with characteristic particle diameter d', is available for use in the prototype. d' will be assumed to be some characteristic measure, such as d_{50}, but this need not be specified for our purposes, as long as the same choice is always used.

The particulate material for use in the model must be prepared from the original material by crushing or sieving, or both. Quantities in each size range must be adjusted to fulfil the requirements of a sample with characteristic particle diameter d'', where $d'' = d' D''/D'$. The size distribution of the model sample should be identical with that of the prototype sample, taking account of the factor (D''/D') in size reduction. In other words, if d_x is any quantile†, such as d_{50} or d_{85}, we must ensure that $d''_x/d'_x = D''/D'$, for any value of x (Figure (B-1)).

Now, we suppose that a double series of runs is carried out in model and prototype, in which particle sizes satisfy

$$Z' = \frac{D'}{d'} = \frac{D''}{d''} = Z'' = Z_1.$$

We also satisfy $C' = C'' = C_1$.

For simplicity we assume that each value of i in a model run is replicated in a prototype run, i.e., $i'_k = i''_k = i_{1_k}$, where subsript k indicates the k-th run in each pipe. (This subscript need not be written explicitly, if understood, and will be omitted to simplify equations.)

For a pair of runs with $i = i_1$, we measure the mean velocities V' and V''. According to Equation (B-18) we should find

$$Y' = \frac{V'}{v'_*} = \frac{V''}{v''_*} = Y''$$

† I.e., a value of d that separates two parts of a size distribution.

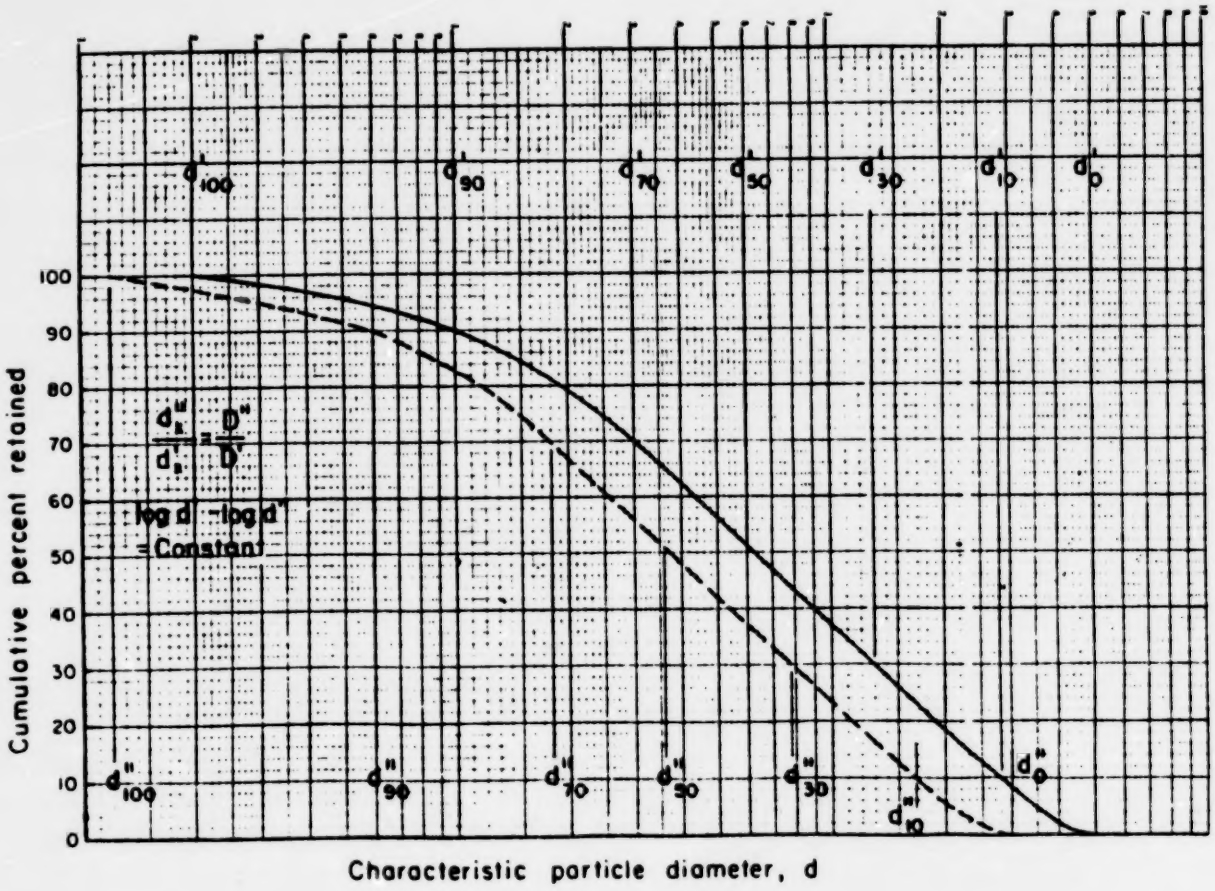

Fig. (B-1) — Particle-size-distribution curve

as we may see as follows:

$$\lambda_{v_*^2} = \lambda_d \quad \text{(as shown before), i.e.} \quad \frac{(v_*^2)''}{(v_*^2)'} = \frac{d''}{d'}$$

therefore,

$$\frac{(v_*^2)'}{d'} = \frac{i_1 g D'/4}{d'} = \frac{i_1 g D''/4}{d''} = \frac{(v_*^2)''}{d''}$$

therefore,

$$Y' = \frac{\rho(v_*^2)'}{\gamma_s d'} = \frac{\rho(v_*^2)''}{\gamma_s d''} = Y''.$$

Thus, Equation (B-18) gives

$$c' = \phi_V(Y', Z', C') = \phi_V(Y'', Z'', C'') = c''$$

or,

$$\Pi'_V = \frac{V'}{v'_*} = \frac{V''}{v''_*} = \Pi''_V$$

and,

$$\frac{V'}{V''} = \frac{v'*}{v''_*} = \sqrt{\frac{d'}{d''}} = \sqrt{\frac{D'}{D''}}.$$

As Z, and C are constants in the series of measurements (with k same, i.e., for constant C and Z, but with varying V), we may plot c against Y. Each value of c is calculated from the measured values of i and V, by the formula

$$c = \frac{V}{v_*} = \frac{V}{\sqrt{i_1 g D/4}}. \qquad Eq\ (B-20)$$

Each value of Y is calculated by

$$Y = \frac{\rho v_*^2}{\gamma_* d} = \frac{\rho i_1 g D/4}{g(\rho_* - \rho)d} = \frac{i_1}{4(s-1)}\left(\frac{D}{d}\right) = \frac{i_1 Z_1}{4(s-1)}. \qquad Eq\ (B-21)$$

When c is plotted against Y, the result will be as shown schematically in Figure (B-2). As full similarity is assumed, the results from both model and prototype will fall on a common line.

But to take a different point of view, if two different plotted lines are found, we may be sure that the prototype and model flows are not similar. The lack of similarity must result from the effects of variables $X = \frac{v_* d}{\nu}$, or $X_6 = k/d$, or both (these were the two variables dropped, according to the initial hypotheses).

If such effects occur, the plots from the double series will appear as shown schematically in Figure (B-3).

Under the conditions of the tests, the differences between the curves are due solely to the differences between X' and X'', and between X'_6 and X''_6. The latter differences can be expressed in terms of (D'/D''), as follows (re. the terms ν and $(D'/D'')^{3/2}$ see Note 1 at end of this section):

$$X' = \frac{v'_* d''}{\nu} = \frac{v'_* D'}{\nu}\frac{d''}{D'} = \frac{v'_*}{\sqrt{D'}}\frac{(D')^{3/2}}{\nu}\frac{d''}{D'} = \frac{v''_*}{\sqrt{D''}}\frac{(D'')^{3/2}}{\nu}\left(\frac{D'}{D''}\right)^{3/2}\frac{d''}{D''} =$$

$$= \frac{v''_* d''}{\nu}\left(\frac{D'}{D''}\right)^{3/2} = X''\left(\frac{D'}{D''}\right)^{3/2} \qquad Eq\ (B-22)$$

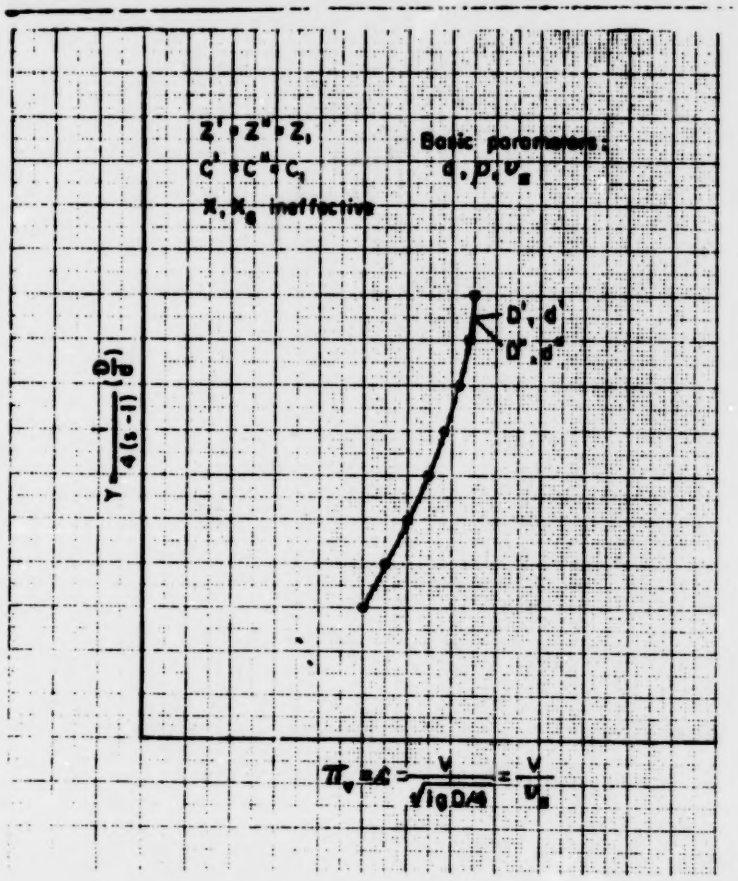

Fig. (B-2) — Similarity between *model* and *prototype*, with fully rough turbulence in both

and,

$$X_6' = \frac{k}{d'} = \frac{k}{d''}\frac{d''}{d'} = \frac{k}{d''}\frac{D''}{D'} = X_6''\left(\frac{D''}{D'}\right). \qquad Eq\ (B-23)$$

* * * * * *

It is now useful to consider the types of results that may follow from choosing a different group of basic characteristic parameters for forming the dimensionless variables of the problem. Previously, the parameters d, ρ, v_* were chosen. As shown by Yalin (1977, p. 62), the critical variables, X and Y, that resulted, have been widely used in studies of two-phase phenomena, under various guises. X is a Reynolds number, and Y is a Froude number, familiar to many.

Let us investigate a new choice of basic parameters, d, ρ, γ_*, a group offering a special advantage emphasized by Yalin. This is also a set of independent parameters in the sense

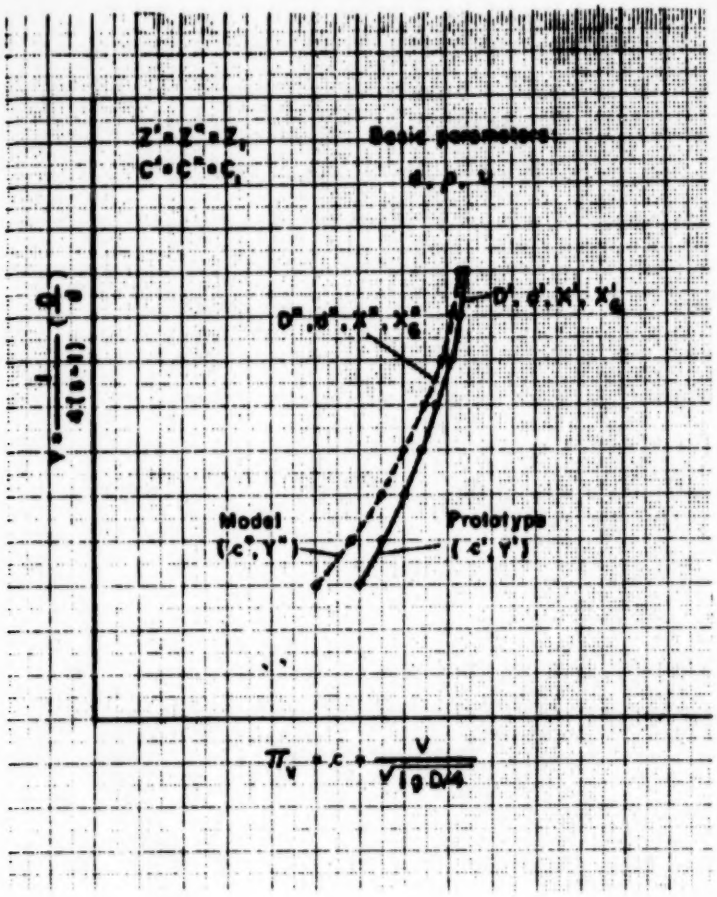

Fig. (B-3) — Dissimilar flows in *model* and *prototype*: viscosity or wall roughness effects, or both

that no product of the form d^a, ρ^b, γ_s^c, can be formed that is dimensionless. Table (B-3) gives the new set of variables, corresponding to the new choice of basic parameters.

It will be seen from Table (B-3) that Z, W, X_5, X_6, X_7, are exactly as before. They involve neither v_* of the former set of basic parameters, nor γ_s of the new set. $Y = \frac{\rho v_*^2}{\gamma_s d}$ occurs as before, but it now represents v_*, whereas it formerly represented γ_s. v_* and γ_s have interchanged rôles.

$X = \frac{v_* d}{\nu} = \frac{v_* d \rho}{\mu}$, which formerly represented μ, is now replaced by $\Xi^{(*)}$, where

$$\Xi = \frac{\gamma_s d^3}{\rho \nu^2} = \frac{\gamma_s d^3 \rho}{\mu^2}.$$

The unique characteristic of Ξ is that it is formed only by the properties of granular material and fluid. (γ_s is classed as a material property, including the effect of gravity). Ξ

(*) See Note 2 at end of this section.

TABLE (B-3)

Dimensionless variables formed with basic parameters d, ρ, γ_s

Characteristic parameters		Variable	
Symbol	Dimensions	Product	Symbol
ρ	ML^{-3}	*basic*	—
μ	$ML^{-1}T^{-1}$	$\frac{\gamma_s d^3}{\rho\nu^2}$	Ξ
ρ_s	ML^{-3}	ρ_s/ρ	W
d	L	*basic*	—
C	1	C	X_5
D	L	D/d	Z
k	L	k/d	X_6
η	1	η	X_7
v_s	LT^{-1}	$\frac{\rho v_s^2}{\gamma_s d}$	Y
γ_s	$ML^{-2}T^{-2}$	*basic*	—

does not contain v_s, and, hence, it is a constant throughout any series of runs in which v_s is varied.

To study any property A of the flow, we must now form the related dimensionless variable as

$$\overline{\overline{\Pi}}_A = d^a \rho^b \gamma_s^c A. \qquad Eq\ (B-24)$$

In the case of the mean velocity, V, we find ‡

$$\overline{\overline{\Pi}}_V = d^{-\frac{1}{2}} \rho^{\frac{1}{2}} \gamma_s^{-\frac{1}{2}} V = \frac{V}{d^{\frac{1}{2}} \rho^{-\frac{1}{2}} [g(\rho_s - \rho)]^{\frac{1}{2}}} = \frac{V}{d^{\frac{1}{2}} g^{\frac{1}{2}} (s-1)^{\frac{1}{2}}} =$$

$$= \frac{V}{\sqrt{dg(s-1)}} = \frac{V}{\sqrt{Dg(s-1)}} \sqrt{\frac{D}{d}}. \qquad Eq\ (B-25)$$

Thus, the equivalent of Equation (B-5) becomes

$$\overline{\overline{\Pi}}_V = \frac{V}{\sqrt{dg(s-1)}} = \overline{\overline{\phi}}_V \left(\Xi,\ Y,\ Z,\ W,\ X_5,\ X_6,\ X_7 \right). \qquad Eq\ (B-26)$$

‡ See Note 3 at end of this section.

This equation expresses exactly the same relationship as was previously expressed by Equation (B-5), but one equation may have practical advantages over the other, which will be indicated later.

* * * * * *

We now return to the double series of measurements intended to test Equation (B-18). The previously stated hypothesis (1) now becomes hypothesis $\overline{\overline{(1)}}$, (on the basis of the set of new parameters d, ρ, γ_s), i.e.,:

$\overline{\overline{(1)}}$ The variable Ξ, representing the direct contribution of μ to energy loss, will be dropped from Equation (B-26).

Then, Equations (B-17) and (B-18) become

$$\overline{\overline{\Pi}}_V = \frac{V}{\sqrt{dg(s-1)}} = \overline{\overline{\phi}}_V\left(\frac{\rho v_*^2}{\gamma_s d},\ \frac{D}{d},\ C\right). \qquad Eq\ (B-27)$$

or,

$$\overline{\overline{\Pi}}_V = \overline{\overline{\phi}}_V\left(Y,\ Z,\ C\right). \qquad Eq\ (B-28)$$

Plotted in terms of $\overline{\overline{\Pi}}_V$ and Y, the same basic data shown in Figure (B-2) (in terms of Π_V and Y) will now appear as shown in Figure (B-4). The first advantage of the new set of basic parameters to be noted is that Figure (B-4) shows a simpler expression of the relation between i and V than does Figure (B-2). $\overline{\overline{\Pi}}_V$ involves only V and other parameters that are constant relative to the run, whereas Π_V also involves i.

Similarly, the data basic to Figure (B-3) will appear as shown in Figure (B-5), when plotted in terms of $\overline{\overline{\Pi}}_V$ and of Y. The second advantage of the new set of basic parameters to be noted is that Ξ' and Ξ'' are each constant in relation to the whole of each of the runs characterized by them. In contrast, X' and X'' are not constants relative to their associated lines in Figure (B-3), since $X = \frac{v_* d}{\nu} = \frac{d\sqrt{igD/4}}{\nu}$.

When experimental results show that the flows in pipes of different sizes differ (under the conditions of particle size and concentration assumed), the following equation is appropriate in contrast to that of Equation (B-27):

$$\overline{\overline{\Pi}}_V = \frac{V}{\sqrt{dg(s-1)}} = \overline{\overline{\phi}}_V\left(\frac{\gamma_s d^3}{\rho\nu^2},\ \frac{\rho v_*^2}{\gamma_s d},\ \frac{D}{d},\ C,\ \frac{k}{d}\right) \qquad Eq\ (B-29)$$

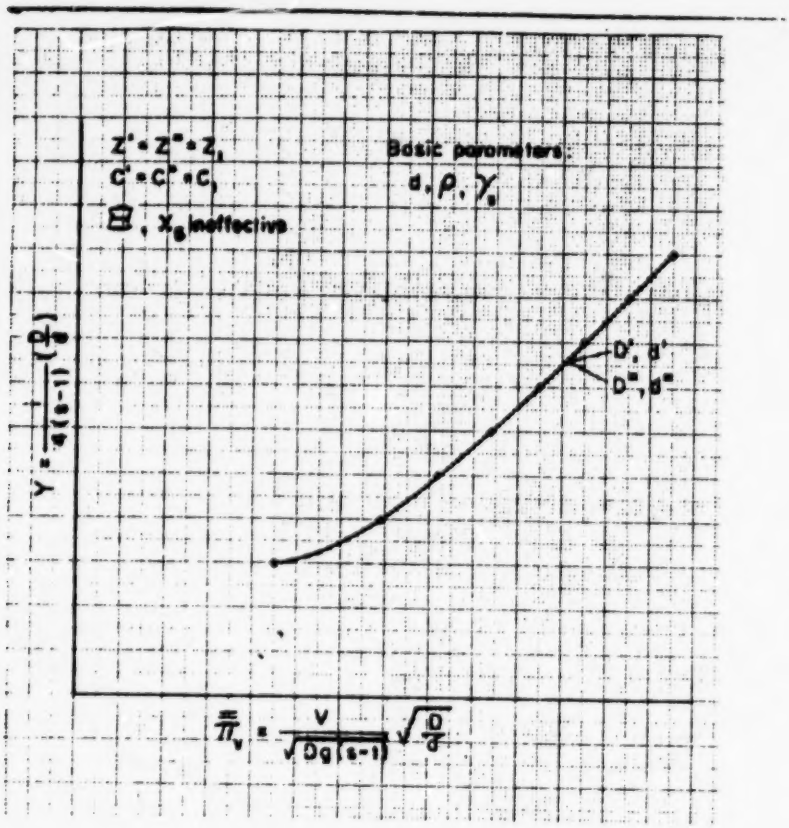

Fig. (B-4) — Data of Figure (B-2) expressed in terms of new basic parameters

i.e.,

$$\overline{\overline{\Pi}}_V = \overline{\overline{\phi}}_V\left(\Xi,\ Y,\ Z,\ X_5,\ X_6\right).$$

For each pipe size, D, all variables are constant, except‡ $\overline{\overline{\Pi}}_V$ and Y.

This should simplify the problem of finding an empirical or theoretical expression to fit the experimental results. An illustration of the practical advantage of using ρ, d, γ_s as basic parameters for forming all dimensionless variables is given in Appendix C of this report.

Note 1

re. Eq (B-22) : (a) the term $\left(\frac{D'}{D''}\right)^{3/2}$ is present because $\frac{d'}{D'} = \frac{d''}{D''}$ is true, and $\frac{v'_*}{\sqrt{D'}} = \frac{v''_*}{\sqrt{D''}}$ is also true; (b) the term ν is a constant for both the *model* and the *prototype*

‡ $\overline{\overline{\Pi}}_V$ is not constant, because V varies; Y is not constant, because i varies in v_*.

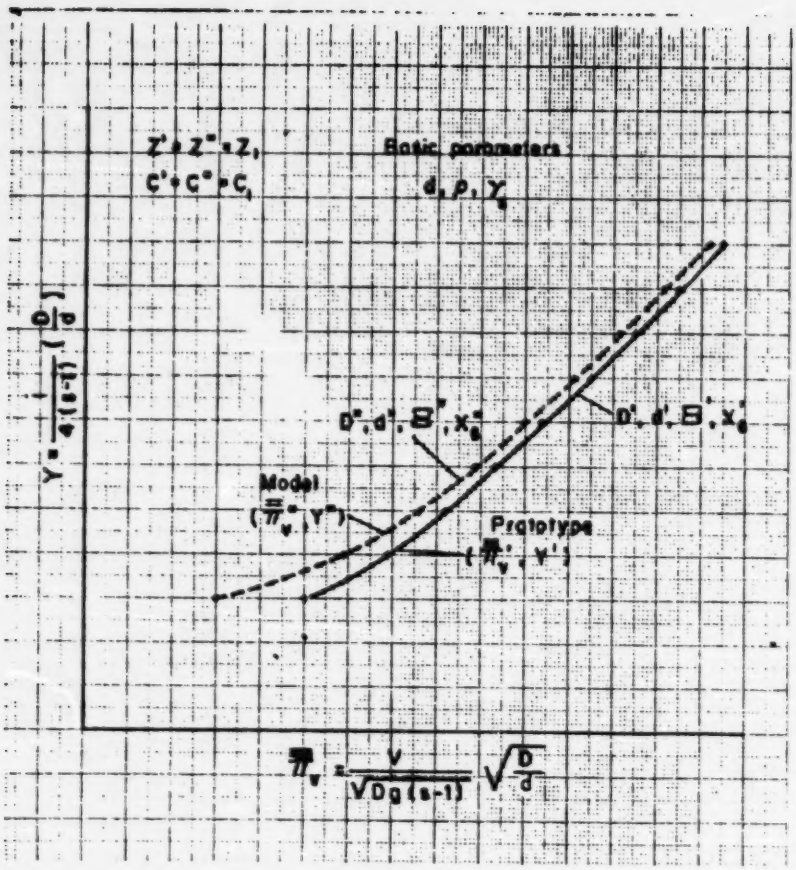

Fig. (B-5) — Data of Figure (B-3) expressed in terms of new basic parameters

Note 2

re. p. (B-120), footnote * : to find Ξ, let $\left[d^a\ \rho^b\ \gamma_s^c\ \mu^{-1}\right] = M^0\ L^0\ T^0 \equiv 1$.

Then, $\left[L^a \left(\frac{M}{L^3}\right)^b \left(\frac{M}{L^2T^2}\right)^c \left(\frac{M}{LT}\right)^{-1}\right] = M^0\ L^0\ T^0$.

Therefore, the equations given by the exponents for L, M, and T, respectively, are:
$(a - 3b - 2c + 1) = 0$; $(b + c - 1) = 0$; and $(-2c + 1) = 0$.
Hence: $c = \frac{1}{2}$; $b = \frac{1}{2}$; and $a = \frac{3}{2}$.

Therefore, the dimensionless product representing Ξ is equal to:

$$\Xi = \left[d^{3/2},\ \rho^{1/2},\ \gamma_s^{1/2},\ \mu^{-1}\right] = \frac{d^{3/2}\,\gamma_s^{1/2}\,\rho^{1/2}}{\mu} = \frac{d^3\,\gamma_s\,\rho}{\mu^2} = \frac{d^3\,\gamma_s}{(\mu^2/\rho^2)\,\rho}.$$

Note 3

re. p. (B-121), footnote ‡: to find $\overline{\overline{\Pi}}_V$, let $\left[d^a\ \rho^b\ \gamma_s^c\ V^1\right] = M^0\ L^0 T^0 \equiv 1$.

Then, $\left[L^a \left(\frac{M}{L^3}\right)^b \left(\frac{M}{L^2T^2}\right)^c \left(\frac{L}{T}\right)^1\right] = M^0\ L^0\ T^0$.

Therefore, the equations given by the exponents for L, M, and T, respectively, are:
$(a - 3b - 2c + 1) = 0$; $(b + c) = 0$; and $(-2c + 1) = 0$.
Hence: $c = -\frac{1}{2}$; $b = \frac{1}{2}$ and $a = -\frac{1}{2}$.
Therefore, the dimensionless product representing $\overline{\overline{\Pi}}_V$ is equal to:

$$\overline{\overline{\Pi}}_V = \left[d^{-1/2},\ \rho^{1/2},\ \gamma_s^{-1/2},\ V\right].$$

APPENDIX C

Discussion of the Relation between the Variable, Ξ, and the Drag Coefficient, C_D

Discussion of the Relation between the Variable $\Xi = \frac{\gamma_s d^3 \rho}{\mu^2} = \frac{\gamma_s d^3}{\rho \nu^2}$, and the Drag Coefficient, C_D

Haas et al. (1980, p. 29), as well as Shook et al. (1981, p. 87), refer to a convenient set of functions, relating experimentally determined values of drag coefficients, C_D, for coal particles, to $C_D R_{ep}^2$, in which:

R_{ep} is $(wd\rho/\mu)$,
d is particle diameter,
w is the terminal fall velocity in water,
ρ is the density of water,
μ is the viscosity of water.

The characteristic parameters of terminal fall velocity in water (far from any boundary) are:

$$\rho,\ \mu,\ \rho_s,\ d,\ g$$

If we replace g by $\gamma_s = g(\rho_s - \rho) = g\rho(s-1)$ (as shown in Appendix B), we may again eliminate ρ_s. Yalin (1977, p. 69) shows that ρ_s does not influence any steady particle velocity involved in a hydraulic flow. ρ_s only appears directly‡ in expressions for particle acceleration.

Thus, the equation giving w in terms of the characteristic parameters is

$$w = f_w(\rho,\ \mu,\ d,\ \gamma_s). \qquad Eq\ (C-1)$$

If ρ, d, and γ_s are chosen as the basic parameters (as was done in Appendix B, to develop Table (B-3)), only one parameter, μ, is left on the right of the equation to form a dimensionless variable with them. This variable is, of course, $\Xi = \frac{\gamma_s d^3}{\rho \nu^2}$.

The dimensionless variable to represent w on the left of the equation is $\overline{\overline{\Pi}}_w = \frac{w}{\sqrt{dg(s-1)}}$. Since w is a velocity, $\overline{\overline{\Pi}}_w$ has the same form as $\overline{\overline{\Pi}}_V$, representing the velocity V in Equation (B-25) (see Appendix B).

‡ I.e., through the variable $W = \rho_s/\rho$.

Thus, the dimensionless equation for w is[§]

$$\overline{\overline{\Pi}}_w = \overline{\overline{\phi}}_w(\Xi) \qquad Eq\ (C-2)$$

or,

$$\overline{\overline{\Pi}}_w = \frac{w}{\sqrt{dg(s-1)}} = \overline{\overline{\phi}}_w\left(\frac{\gamma_s d^3}{\rho\nu^2}\right). \qquad Eq\ (C-3)$$

Now, the usual formula for C_D is, with k as a constant,

$$C_D = k\frac{gd}{w^2}\frac{(\rho_s-\rho)}{\rho} = k\frac{d\,g\,(s-1)}{w^2} = k\frac{\gamma_s d}{\rho w^2} = k\frac{1}{\left(\overline{\overline{\Pi}}_w\right)^2}.$$

Hence,

$$C_D = \overline{\overline{\phi'}}_w\left(\frac{\gamma_s d^3}{\rho\nu^2}\right) \qquad Eq\ (C-4)$$

where,

$$\overline{\overline{\phi'}}_w\left(\Xi\right) = k\frac{1}{\left(\overline{\overline{\Pi}}_w\right)^2} = k\frac{1}{\left[\overline{\overline{\phi}}_w\left(\frac{\gamma_s d^3}{\rho\nu^2}\right)\right]^2}.$$

Further,

$$C_D Re_p^2 = k\frac{gd}{w^2}\frac{(\rho_s-\rho)}{\rho}\left(\frac{wd\rho}{\mu}\right)^2 = k\frac{g(\rho_s-\rho)d^3}{\rho\nu^2} = k\frac{\gamma_s d^3}{\rho\nu^2} = k\,\Xi$$

where, $k = \frac{4}{3}$, [e.g., see Haas et al. (1980, p. 29)].

Thus, Equation (C-4) can be written as

$$C_D = \overline{\overline{\phi'}}_w\left(\frac{1}{k}\,C_D\,R_{ep}^2\right). \qquad Eq\ (C-5)$$

This is the relationship between C_D and $C_D R_{ep}^2$ referred to above. The set of functions given by Shook et al. (1981) are the empirical approximate expressions for Equation (C-5) in given ranges of $C_D R_{ep}^2$, or equivalently in ranges of Ξ.

The use of equation (C-5) is completely equivalent to Equation (C-3) in the standard form developed by Yalin (1977). Equation (C-5), in a simpler situation, is analogous to Equation (B-29), referred to in Appendix B. It is reasonable to expect that the use of the

§ In equation (B-26), $\overline{\overline{\Pi}}_V$, is a function of Ξ and several other variables. The latter are not present here, because the parameters C, D, k, η, and v_s are now missing.

basic parameters ρ, d, and γ_s, will also be advantageous in the more complex situation of slurry flow. Primarily, it may assist the development of empirical expressions from systematically obtained data. It should also facilitate theoretical analysis.

Important limitations, regarding the specification of parameters, can also be illustrated here. In equation (C-5) coal particles are represented only by d. Shape parameters are omitted. Strictly speaking, Equation (C-5) must be used only in connection with coal particles having the same shape parameters possessed by the particles that provided the original data.

In addition, the measurement of d must observe the same conventions as used in measuring d for the original data. These conventions, and the unspecified shape parameters, are built into the resulting relations expressing Equation (C-5).

If Equation (C-5) is used in connection with particles that may differ in shape from the particles for which the equation was developed, erroneous results may be given. Tests on the particles in question are required before the order of such errors can be known accurately.